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INTERPOLATION IN TERM FUNCTOR LOGIC

J.-MARTÍN CASTRO-MANZANO
UPAEP University
Faculty of Philosophy
Mexico
josemartin.castro@upaep.mx
https://orcid.org/0000-0003-2227-921X

SUMMARY: Given some links between Lyndon's Interpolation Theorem, term distribution, and Sommers and Englebretsen's logic, in this contribution we attempt to capture a sense of interpolation for Sommers and Englebretsen's Term Functor Logic. In order to reach this goal we first expound the basics of Term Functor Logic, together with a sense of term distribution, and then we offer a proof of our main contribution.

KEY WORDS: term logic, Aristotelian logic, syllogistic, distribution, Craig-Lyndon interpolation

RESUMEN: Dados algunos vínculos entre el Teorema de Interpolación de Lyndon, la distribución de términos y la lógica de Sommers y Englebretsen, en esta contribución intentamos capturar un sentido de interpolación para la Lógica de Términos y Funtores de Sommers y Englebretsen. Para alcanzar este objetivo, primero exponemos los conceptos básicos de la Lógica de Términos y Funtores, junto con un sentido de distribución de términos, y luego ofrecemos una prueba de nuestra contribución.

PALABRAS CLAVE: lógica de términos, lógica aristotélica, silogística, distribución, interpolación de Craig-Lyndon

1. *Introduction*

Broadly construed, Craig's Interpolation Theorem (1957) states that for any pair of formulas ϕ and ψ such that ϕ implies ψ, an interpolant is a formula γ such that ϕ implies γ, γ implies ψ, and the non-logical symbols in γ occur both in ϕ and in ψ. In other words, if we let $Rel(\phi)$ stand for the set of relation symbols in ϕ (resp. $Rel(\psi)$), then Craig's theorem would say what follows: if ϕ and ψ are sentences such that $\vdash \phi \Rightarrow \psi$, then there exists a sentence γ (an interpolant) such that $\vdash \phi \Rightarrow \gamma$, $\vdash \gamma \Rightarrow \psi$, and $Rel(\gamma) \subseteq Rel(\phi) \cap Rel(\psi)$.

Lyndon gave a generalization of Craig's Interpolation Theorem in terms of the polarities of the relation symbols involved in a formula (1959). His theorem states that if ϕ and ψ are first order formulas, and ϕ implies ψ, then there exists a formula γ (an interpolant) such that ϕ implies γ, γ implies ψ, and every relation symbol which occurs positively (resp. negatively) in γ occurs positively (resp. negatively) both in ϕ and ψ. In other words, if we let $Rel^{+}(\phi)$ (resp. $Rel^{-}(\phi)$) stand for the set of relation symbols with at least one positive (resp. at least one negative) occurrence in ϕ, then, if $\vdash \phi \Rightarrow \psi$, then there is an interpolant γ with respect to Rel^{+} and Rel^{-}.

Now, Hodges has argued, rather convincingly, that Lyndon's Interpolation Theorem gives a generalization of the laws of distribution for traditional, Aristotelian syllogistic (1998), which is a claim we find quite interesting because distribution —a concept we use when we talk about terms that appear under the scope of a universal quantifier (Keynes 1906; Sommers 1975; Wilson 1987)— serves a fundamental purpose in Sommers and Englebretsen's Term Functor Logic —a novel logic that recovers some insights of the traditional, Aristotelian logic— in so far as it helps account for a notion of validity (Sommers 1982; Englebretsen 1987; Englebretsen 1996; Sommers and Englebretsen 2000; Englebretsen and Sayward 2011).

Given these preliminaries, we cannot help but wonder: granted these superficial links between Lyndon's theorem, distribution, and Sommers and Englebretsen's logic, is there some sort of interpolation for said logic? In this contribution we explore this issue and we suggest there is a plausible sense in which we could account for some sort of term interpolation for Sommers and Englebretsen's Term Functor Logic. In order to reach this goal we first expound the basics of Term Functor Logic, together with a sense of term distribution, and then we offer a proof of our main contribution; at the end, we close with a brief discussion.

2. *Term Functor Logic*

Assertoric syllogistic —the logic at the core of traditional, Aristotelian logic— is a term logic that makes use of categorical statements in order to capture a basic notion of assertion. A categorical statement is a statement composed by two terms, a quantity, and a quality. Typically, we say a categorical statement is a statement of the form:

$$<\mathit{Quantity}> < \mathsf{S} > <\mathit{Quality}> < \mathsf{P} >$$

where *Quantity* = {All, Some}, *Quality* = {is (are), is not (are not)}, and S and P are term-schemes, so that we obtain four kinds of categorical statements: the universal affirmative, the universal negative, the particular affirmative, and the particular negative. The following are examples of categorical statements, in said order:

1. All logicians are smart.
2. No logician is smart (i.e., all logicians are not smart).
3. Some logicians are smart.
4. Some logicians are not smart.

From the standpoint of Sommers and Englebretsen's Term Functor Logic (TFL, for short), we say a categorical statement in TFL is a statement of the form:

$$\pm \mathsf{S} \pm \mathsf{P}$$

where $\pm$ is shorthand for the $+$ and $-$ functors, and S and P are term-schemes. So, for example, we can model the four traditional, categorical statements in TFL as follows, where the term L stands for *logicians*, and S stands for *smart*:[1]

1. $-\mathsf{L} + \mathsf{S}$
2. $-\mathsf{L} - \mathsf{S}$
3. $+\mathsf{L} + \mathsf{S}$
4. $+\mathsf{L} - \mathsf{S}$

Given this language (i.e., $\mathcal{L}_{TFL} = <\mathcal{T}, \pm>$, where $\mathcal{T} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \ldots\}$ is a set of terms, and $\pm$ is shorthand for the $+$ and $-$ functors), TFL offers a basic sense of validity as follows (Englebretsen 1996, p. 167): a syllogism is valid (in TFL) iff (1) the algebraic sum of the premises is equal to the conclusion, and (2) the number of particular conclusions (*viz.*, zero or one) is equal to the number of particular premises. And so, with this logic we can model assertoric inferences like the one shown in Table 1.

[1] In this context, terms are those elements into which a statement can be divided, that is, into that which is predicated of something (i.e., the predicate) and that of which something is predicated (i.e., the subject), as Aristotle suggested (*Pr. An.* A1, 24b16–17); whereas functors are logical expressions. Terms are what the medieval scholastic philosophers called *categoremata*; whereas functors are *syncategoremata*, that is, words that are not terms but are used to turn terms into more complex terms. As Englebretsen (1996, 2013) explains, within TFL a term might be formed by the use of a single word or a complex of words. In English, for example, *smart*, and *logician* are terms, as well, as *taught Plato*, or *is Greek* are terms; whereas, for example, *and*, *or*, *only if*, *if. . . then*, *all*, *some*, *not*, *is*, and *is not* are functors. This is similar to our current, classical distinction between logical variables and logical constants.

Statement	TFL
1. All philosophers are smart.	$-\mathsf{P}+\mathsf{S}$
2. All logicians are philosophers.	$-\mathsf{L}+\mathsf{P}$
$\vdash$ All logicians are smart.	$-\mathsf{L}+\mathsf{S}$

Table 1. A valid assertoric inference

In this example we can clearly see how the previous definition works: (1) if we add up the premises we obtain the algebraic expression $(-\mathsf{P}+\mathsf{S})+(-\mathsf{L}+\mathsf{P}) = -\mathsf{P}+\mathsf{S}-\mathsf{L}+\mathsf{P} = -\mathsf{L}+\mathsf{S}$, so that the sum of the premises is algebraically equal to the conclusion, and the conclusion is $-\mathsf{L}+\mathsf{S}$, rather than $+\mathsf{S}-\mathsf{L}$, because (2) the number of conclusions with particular quantity (zero in this case) is the same as the number of premises with particular quantity (zero in this case).[2]

The previous definition of validity is an algebraic rendition of the *dictum de omni et nullo* (DON, for short), the principle that states that everything that is affirmed (resp. denied) of a whole can be affirmed (resp. denied) of a part (*Pr. An.* A1, 24b26–30), and so, alternatively, using the concept of distribution, we can define the same notion of validity as follows (Sommers 1989): let $\phi(+\mathsf{M})$ stand for a statement with a positive occurrence of the term M (i.e., M is undistributed) in a formula ϕ, and let $\psi(-\mathsf{M})$ stand for a statement in which M occurs negatively (i.e., M is distributed) in a formula ψ, then the rule would state that if $\phi(-\mathsf{M})$ and $\psi(+\mathsf{M})$, then $\phi(\psi)$, since $-\mathsf{M}$ and $+\mathsf{M}$ cancel each other out. In the previous example we can clearly see the term P occurs distributed in premise 1 (i.e., $-\mathsf{P}$), and undistributed in premise 2 (i.e., $+\mathsf{P}$), and so both occurrences cancel each other out, thus allowing the conclusion (i.e., $-\mathsf{L}+\mathsf{S}$).

This second account of validity, although equivalent to the first definition, depends upon a sense of term distribution defined as follows (Sommers 1989): given a term T in a categorical statement ϕ, we say:

— T is distributed in ϕ iff ϕ is a statement of the form "every T is . . . ;"

— T is undistributed in ϕ iff T is distributed in the contradictory of ϕ.

[2] Although we are exemplifying this logic with syllogistic inferences, this system is capable of representing relational, singular, and compound inferences with ease and clarity. Furthermore, TFL is arguably more expressive than classical first order logic (Englebretsen 1996, p. 172).

Thus, formally, given any statement, its terms are distributed (resp. undistributed) if and only if said terms occur next to the minus functor (resp. next to the plus functor). Informally, given its quantity and its quality, we say a statement distributes its terms if and only if they have a universal quantity or a negative quality. For example, if we recall the previous categorical statements we can pinpoint which terms are distributed (–) and which terms are not (+):

1. All logicians^{-} are smart^{+}.
2. No logician^{-} is smart^{-}.
3. Some logicians^{+} are smart^{+}.
4. Some logicians^{+} are not smart^{-}.

Now, up to this point, given the previous exposition, one could think the notion of validity for this logic only covers monadic or syllogism-like inferences, but that would be a hasty conclusion. We can extend the previous notion of validity in order to cover relational, singular, and compound inferences by noticing that any categorical statement is a well-formed formula of TFL, and either by enlarging the rules of inference (as in Englebretsen 1996) or by implementing tableaux proof methods (as in Castro-Manzano 2018; Castro-Manzano and Reyes-Cárdenas 2018; Castro-Manzano 2020) without losing the notion of term distribution. So, we can easily extend the notion of validity for this logic as follows:

Definition 1 (Valid inference (in TFL)). Let ϕ and ψ stand for arbitrary TFL well-formed formulas. An inference is valid (in TFL), namely, $\vdash -\phi + \psi$ iff ψ is obtained from ϕ by applying some adequate rule of inference (or by following a tableaux procedure).[3]

In other words, ϕ implies ψ if and only ψ is obtained from ϕ by applying some adequate rule of inference (like DON) or by following a tableaux procedure. As an example, and in order to close this section, consider a relational inference *à la* De Morgan in Table 2.

[3] At this point it must be mentioned that TFL admits a Deduction Theorem so that this definition not only applies to cases of conditional logical truths, namely, $\Gamma \cup + \phi \vdash +\psi$ if and only if $\Gamma \vdash -\phi + \psi$, where Γ is a (possibly empty) set of terms. Here is a quick sketch of proof: from left to right, we have $+\psi$ by hypothesis, since $\Gamma \cup + \phi \vdash +\psi$, and given that $-\psi+(-\phi + \psi)$ is a tautology (for $-(-\psi+(-\phi + \psi))$ is a contradiction), then $-\phi + \psi$ follows by DON (i.e., $-\psi+(-\phi + \psi)\ +\psi\ =\ +(-\phi + \psi)\ =\ -\phi + \psi$). Now, from right to left we have $-\phi + \psi$ by hypothesis, given that $\Gamma \vdash -\phi + \psi$, and since $+\phi$ is given by the assumption that $\Gamma \cup + \phi$, it follows that $+\psi$ by DON (i.e., $-\phi+\psi + \phi = +\psi$).

Statement	TFL	
1. Every circle is a figure.	$-\mathsf{C}+\mathsf{F}$	
$\vdash$Whoever draws a circle draws a figure.	$-(+\mathsf{D}+\mathsf{C})+(+\mathsf{D}+\mathsf{F})$	
2. Suppose someone draws a circle.	$+\mathsf{D}+\mathsf{C}$	Assumption
3. Someone draws a figure.	$+\mathsf{D}+\mathsf{F}$	DON from 1 and 2
4. Whoever draws a circle draws a figure.	$-(+\mathsf{D}+\mathsf{C})+(+\mathsf{D}+\mathsf{F})$	Assumption closure

Table 2. A valid relational inference in TFL

To wrap this up, and as we mentioned previously, Hodges has argued that Lyndon's Interpolation Theorem gives a generalization of the laws of distribution for syllogistic (1998), which is something we find interesting because distribution helps account for the notion of validity for TFL, which stems from syllogistic (Sommers 1982; Englebretsen 1987; Englebretsen 1996; Sommers and Englebretsen 2000; Englebretsen and Sayward 2011). Our objective now is to formally explore a sense of term interpolation for TFL.

3. *Interpolation in Term Logic*

Before we offer a formal result regarding term interpolation, let us consider some preliminaries. First, consider that the expression $-\phi+\phi$ is a valid formula or a tautology, and that its negation, $-(-\phi+\phi)=+\phi-\phi$, is a contradiction (Englebretsen 1996). Second, let us define a function *Dist*(ϕ) (resp. *unDist*(ϕ)) that returns the set of distributed terms (resp. undistributed terms) in a categorical statement ϕ. For example, if $\phi := -\mathsf{L}+\mathsf{S}$ then *Dist*(ϕ) $=\{\mathsf{L}\}$ and *unDist*(ϕ) $=\{\mathsf{S}\}$. In particular, in case ϕ is a tautology or a contradiction, we assign *Dist*(ϕ) $=\varnothing$, and *unDist*(ϕ) $=\varnothing$, for any term cannot be both distributed and undistributed. Third, we use the following sort of diagrams as to indicate what follows from what in a given inference:

$$\vdash -\phi+\psi \begin{cases} \phi & \begin{cases} -\mathsf{A}+\mathsf{B} \\ -\mathsf{C}+\mathsf{A} \end{cases} \\ \psi & \begin{cases} -\mathsf{C}+\mathsf{B} \end{cases} \end{cases}$$

So, in this example we are saying that ϕ implies ψ, and that ϕ is the set of premises $\{-\mathsf{A}+\mathsf{B}, -\mathsf{C}+\mathsf{A}\}$, while ψ is the conclusion $-\mathsf{C}+\mathsf{B}$.

Fourth, besides the DON inference rule, TFL includes the simplification rule (*Simp*) as follows: either conjunct can be deduced from a conjunctive formula; from a particularly quantified formula with a conjunctive subject-term, we can deduce either the statement form of the subject-term or a new statement just like the original but without one of the conjuncts of the subject-term (i.e., from $+(+\mathsf{X}+\mathsf{Y})\pm\mathsf{Z}$ we can deduce any of the following: $+\mathsf{X}+\mathsf{Y}$, $+\mathsf{X}\pm\mathsf{Z}$, or $+\mathsf{Y}\pm\mathsf{Z}$), and from a universally quantified formula with a conjunctive predicate-term we can deduce a new statement just like the original but without one of the conjuncts of the predicate-term (i.e., from $-\mathsf{X}\pm(+\mathsf{Y}+\mathsf{Z})$ we can deduce either $-\mathsf{X}\pm\mathsf{Y}$ or $-\mathsf{X}\pm\mathsf{Z}$).

Finally, there is the addition rule (*Add*): any two previous formulae in a sequence can be conjoined to yield a new formula, and from any pair of previous formulae that are both universal affirmations and share a common subject-term a new formula can be derived that is a universal affirmation, has the subject-term of the previous formulae, and has the conjunction of the predicate-terms of the previous formulae as its predicate-term (i.e., from $-\mathsf{X}+\mathsf{Y}$ and $-\mathsf{X}+\mathsf{Z}$ we can deduce $-\mathsf{X}+(+\mathsf{Y}+\mathsf{Z})$).

Given these preliminaries, we now suggest a notion of term interpolation for TFL as follows:

Proposition 1. (Term interpolation). Let ϕ, ψ, and γ stand for arbitrary TFL well-formed formulas. If ϕ implies ψ, i.e., if $\vdash -\phi+\psi$, then there is an interpolant γ s.t. $\vdash -\phi+\gamma$, $\vdash -\gamma+\psi$, and every term that occurs (un)distributed in γ occurs (un)distributed in both ϕ and ψ.

Proof. We proceed by cases. (Case 1) So, first, suppose $\vdash -\phi+\psi$, and let $Dist(\phi)\cap Dist(\psi) = unDist(\phi)\cap unDist(\psi) = \varnothing$. In this case, we have four alternatives:

1. Both ϕ and ψ are tautologies with no terms in common, i.e., $\phi := -\mathsf{T}+\mathsf{T}$, and $\psi := -\mathsf{T}'+\mathsf{T}'$ s.t. $\mathsf{T}\neq\mathsf{T}'$ for arbitrary well-formed formulas T and T'. In this case, any other tautology γ with no common terms is an interpolant, for $unDist(\gamma)\subseteq unDist(\phi)\cap unDist(\psi)=\varnothing$, and $Dist(\gamma)\subseteq Dist(\phi)\cap Dist(\psi)=\varnothing$.

2. If ψ is a tautology and ϕ is empty, i.e., $\phi:=\varnothing$, and $\psi := -\mathsf{T}+\mathsf{T}$. In this case, any other tautology γ with no common terms is an interpolant, for $unDist(\gamma)\subseteq unDist(\phi)\cap unDist(\psi)=\varnothing$, and $Dist(\gamma)\subseteq Dist(\phi)\cap Dist(\psi)=\varnothing$.

3. If ψ is a tautology and ϕ is an arbitrary formula with no terms in common, i.e., $\phi := \pm\mathsf{T}$, and $\psi := -\mathsf{T} + \mathsf{T}$ s.t. $\mathsf{T} \neq \mathsf{T}'$. In this case, any other tautology γ with no common terms is an interpolant, for $unDist(\gamma) \subseteq unDist(\phi) \cap unDist(\psi) = \varnothing$, and $Dist(\gamma) \subseteq Dist(\phi) \cap Dist(\psi) = \varnothing$.

4. If ϕ is a contradiction and ψ is any formula with no common terms with ϕ, i.e., $\phi := +\mathsf{T} - \mathsf{T}$, and $\psi := \pm\mathsf{T}'$ s.t. $\mathsf{T} \neq \mathsf{T}'$. In this case, any other contradiction γ with no common terms is an interpolant, for $unDist(\gamma) \subseteq unDist(\phi) \cap unDist(\psi) = \varnothing$, and $Dist(\gamma) \subseteq Dist(\phi) \cap Dist(\psi) = \varnothing$.

(Case 2) Second, in case $Dist(\phi) \cap Dist(\psi) \neq \varnothing$ and $unDist(\phi) \cap unDist(\psi) \neq \varnothing$, we have to proceed by induction on the size of such sets. So, for the base case, consider $Dist(\phi) - Dist(\psi) = \varnothing$ and $unDist(\phi) - unDist(\psi) = \varnothing$. In this case, all terms of ϕ are terms of ψ, both distributed and undistributed. Then, $\phi = \gamma$ is a proper interpolant because $\vdash -\phi + \phi$ and $\vdash -\phi + \psi$ (by assumption), so that $\vdash -\phi + \psi$ follows by applying DON. Now, for the inductive case, consider $Dist(\phi) - Dist(\psi) = j$ and $unDist(\phi) - unDist(\psi) = k$. In this case, since $\vdash -\phi + \psi$, there are some common (un)distributed terms between ϕ and ψ. Fix some γ in such a way that $Dist(\gamma) \subseteq Dist(\phi) \cap Dist(\psi)$ and $unDist(\gamma) \subseteq unDist(\phi) \cap unDist(\psi)$. We have to show that $\vdash -\phi + \gamma$. Let $\mathbb{T}^1_j$ stand for a sequence of terms $\pm\mathsf{T}_1, \ldots, \pm\mathsf{T}_j$. Then we have three general alternatives in which this condition holds by *Simp*:

$$Alt.1. \vdash -\phi + \gamma \begin{cases} \phi & \begin{cases} +\mathbb{T}^1_j \pm \mathbb{T}^{j+1}_k \end{cases} \\ \gamma & \begin{cases} +\mathbb{T}^1_j \end{cases} \end{cases}$$

$$Alt.2. \vdash -\phi + \gamma \begin{cases} \phi & \begin{cases} +\mathbb{T}^1_j \pm \mathbb{T}^{j+1}_k \end{cases} \\ \gamma & \begin{cases} \pm\mathbb{T}^{j+1}_k \end{cases} \end{cases}$$

$$Alt.3. \vdash -\phi + \gamma \begin{cases} \phi & \begin{cases} -\mathbb{T}^1_j \pm (+\mathbb{T}^{j+1}_k + \mathbb{T}^{k+1}_m) \end{cases} \\ \gamma & \begin{cases} -\mathbb{T}^1_j \pm \mathbb{T}^{k+1}_m \end{cases} \end{cases}$$

But in all these alternatives, we can deduce ψ from γ, i.e., $\vdash -\gamma + \psi$, in such a way that every term (un)distributed in γ is also (un)distributed in ψ:

$$Alt.1. \vdash -\gamma + \psi \begin{cases} \gamma & \begin{cases} +\mathbb{T}^1_j \end{cases} \\ \psi & \begin{cases} - - \mathbb{T}^1_j - -\mathbb{T}^{j+1}_k \end{cases} \end{cases}$$

$$Alt.2. \vdash -\gamma + \psi \begin{cases} \gamma & \begin{cases} \pm\mathbb{T}^{j+1}_k \end{cases} \\ \psi & \begin{cases} - - (\pm\mathbb{T}^1_j) - -\mathbb{T}^{j+1}_k \end{cases} \end{cases}$$

$$Alt.3. \vdash -\gamma + \psi \begin{cases} \gamma & \begin{cases} -\mathbb{T}^1_j \pm \mathbb{T}^{k+1}_m \end{cases} \\ \psi & \begin{cases} -\mathbb{T}^1_j \pm \mathbb{T}^{k+1}_m \end{cases} \end{cases}$$

In alternative 1, ψ follows from γ by *reductio*, for suppose ψ is not the case. Then we would get $+ - \mathbb{T}^1_j + -\mathbb{T}^{j+1}_k$, but then by *Simp* we obtain $-\mathbb{T}^1_j$, which contradicts γ. In alternative 2, ψ follows from γ by a similar procedure. Last, in alternative 3 ψ follows from γ as in the base case. But in all these alternatives ψ follows from γ and γ is such that $Dist(\gamma) \subseteq Dist(\phi) \cap Dist(\psi)$ and $unDist(\gamma) \subseteq unDist(\phi) \cap unDist(\psi)$.

Now, for the inductive case in which $Dist(\phi) - Dist(\psi) = j{+}1$ and $unDist(\phi) - unDist(\psi) = k{+}1$, consider the next alternatives:

$$Alt.1'. \vdash -\phi + \gamma \begin{cases} \phi & \begin{cases} +\mathbb{T}^1_{j+1} \pm \mathbb{T}^{j+2}_{k+1} \end{cases} \\ \gamma & \begin{cases} +\mathbb{T}^1_{j+1} \end{cases} \end{cases}$$

$$Alt.2'. \vdash -\phi + \gamma \begin{cases} \phi & \begin{cases} +\mathbb{T}^1_{j+1} \pm \mathbb{T}^{j+2}_{k+1} \end{cases} \\ \gamma & \begin{cases} \pm\mathbb{T}^{j+2}_{k+1} \end{cases} \end{cases}$$

$$Alt.3'. \vdash -\phi + \gamma \begin{cases} \phi & \begin{cases} -\mathbb{T}^1_{j+1} \pm (+\mathbb{T}^{j+2}_{k+1} + \mathbb{T}^{j+2}_{k+1} + \mathbb{T}^{k+2}_m) \end{cases} \\ \gamma & \begin{cases} -\mathbb{T}^1_{j+1} \pm (+\mathbb{T}^{j+2}_{k+1} + \mathbb{T}^{k+2}_m) \end{cases} \end{cases}$$

But again, in all these cases, $\vdash -\gamma + \psi$, as follows:

$$Alt.1'. \vdash -\gamma + \psi \begin{cases} \gamma & \begin{cases} +\mathbb{T}^1_{j+1} \end{cases} \\ \psi & \begin{cases} --\mathbb{T}^1_{j+1} - -\mathbb{T}^{j+2}_{k+1} \end{cases} \end{cases}$$

$$Alt.2'. \vdash -\gamma + \psi \begin{cases} \gamma & \begin{cases} \pm\mathbb{T}^{j+2}_{k+1} \end{cases} \\ \psi & \begin{cases} --(\pm\mathbb{T})^{j+2}_{k+1} - -\mathbb{T}^{k+2}_{m} \end{cases} \end{cases}$$

$$Alt.3'. \vdash -\gamma + \psi \begin{cases} \gamma & \begin{cases} -\mathbb{T}^1_{j+1} \pm (+\mathbb{T}^{j+2}_{k+1} + \mathbb{T}^{k+2}_{m}) \end{cases} \\ \psi & \begin{cases} -\mathbb{T}^1_{j+1} \pm (+\mathbb{T}^{j+2}_{k+1} + \mathbb{T}^{k+2}_{m}) \end{cases} \end{cases}$$

(Case 3) Third, in case $Dist(\phi) \cap Dist(\psi) = \varnothing$ and $unDist(\phi) \cap unDist(\psi) \neq \varnothing$, let us use induction on the size of such sets. For the base case, consider $Dist(\phi) - Dist(\psi) = \varnothing$ and $unDist(\phi) - unDist(\psi) = \varnothing$. In this case, all terms of ϕ are terms of ψ, both distributed and undistributed. For the inductive case, consider $Dist(\phi) - Dist(\psi) = \varnothing$ and $unDist(\phi) - unDist(\psi) = k$. In this case, since $\vdash -\phi+\psi$, there are some common undistributed terms between ϕ and ψ. Fix some γ in such a way that $unDist(\gamma) \subseteq unDist(\phi) \cap unDist(\psi)$. We have to show that $\vdash -\phi + \gamma$. Then we have one alternative:

$$\vdash -\phi + \psi \begin{cases} \phi & \begin{cases} -\mathbb{T}^1_j \pm \mathbb{T}^{j+1}_k \\ +\mathbb{T}^1_j \end{cases} \\ \psi & \begin{cases} --\mathbb{T}^1_j - -(\pm\mathbb{T})^{j+1}_k \end{cases} \end{cases}$$

That ψ follows from ϕ is clear by *reductio*, for suppose ψ is not the case. Then we obtain $+-\mathbb{T}^1_j + -(\pm\mathbb{T})^{j+1}_k$, and by *Simp*, we get $\mp\mathbb{T}^{j+1}_k$. But by applying DON to ϕ we get $\pm\mathbb{T}^{j+1}_k$, which is a contradiction.

But now, $\vdash -\phi+\gamma$ (by DON), and $\vdash -\gamma+\psi$ by *reductio* as follows:

$$\vdash -\phi + \gamma \begin{cases} \phi & \begin{cases} -\mathbb{T}^1_j \pm \mathbb{T}^{j+1}_k \\ +\mathbb{T}^1_j \end{cases} \\ \gamma & \begin{cases} \pm\mathbb{T}^{j+1}_k \end{cases} \end{cases}$$

$$\vdash -\gamma + \psi \begin{cases} \gamma & \begin{cases} \pm\mathbb{T}^{j+1}_k \end{cases} \\ \psi & \begin{cases} --\mathbb{T}^1_j - -\mathbb{T}^{j+1}_k \end{cases} \end{cases}$$

Then clearly, for $unDist(\phi) - unDist(\psi) = k + 1$, consider the next alternative:

$$\vdash -\phi + \psi \begin{cases} \phi & \begin{cases} -\mathbb{T}^1_j \pm \mathbb{T}^{j+1}_{k+1} \\ +\mathbb{T}^1_j \end{cases} \\ \psi & \begin{cases} - - \mathbb{T}^1_j - -\mathbb{T}^{j+1}_{k+1} \end{cases} \end{cases}$$

And thus $\vdash -\phi + \gamma$, and $\vdash -\gamma + \psi$ as follows:

$$\vdash -\phi + \gamma \begin{cases} \phi & \begin{cases} -\mathbb{T}^1_j \pm \mathbb{T}^{j+1}_{k+1} \\ +\mathbb{T}^1_j \end{cases} \\ \gamma & \begin{cases} \pm\mathbb{T}^{j+1}_{k+1} \end{cases} \end{cases}$$

$$\vdash -\gamma + \psi \begin{cases} \gamma & \begin{cases} \pm\mathbb{T}^{j+1}_{k+1} \end{cases} \\ \psi & \begin{cases} - - \mathbb{T}^1_j - -\mathbb{T}^{j+1}_{k+1} \end{cases} \end{cases}$$

(Case 4) Finally, the remaining case is similar to the third case, *mutatis mutandis*. $\square$

To further illustrate this result, let us consider some examples.

Example 1 (A propositional case).[4] Let $\phi := \{-[\mathsf{p}] + [\mathsf{q}], +[\mathsf{p}]\}$ and $\psi := \{-- [\mathsf{r}] --[\mathsf{q}]\}$:

$$\vdash -\phi + \psi \begin{cases} \phi & \begin{cases} -[\mathsf{p}] + [\mathsf{q}] \\ +[\mathsf{p}] \end{cases} \\ \psi & \begin{cases} - - [\mathsf{r}] - -[\mathsf{q}] \end{cases} \end{cases}$$

Since $\vdash -\phi + \psi$, there is an interpolant γ such that $unDist(\gamma) \subseteq unDist(\phi) \cap unDist(\psi) = \{[\mathsf{q}]\}$, and $Dist(\gamma) \subseteq Dist(\phi) \cap Dist(\psi) = \varnothing$, as follows:

$$\vdash -\phi + \gamma \begin{cases} \phi & \begin{cases} -[\mathsf{p}] + [\mathsf{q}] \\ +[\mathsf{p}] \end{cases} \\ \gamma & \begin{cases} +[\mathsf{q}] \end{cases} \end{cases}$$

[4] Compound propositions can be represented in TFL as follows: let P, Q, and R stand for propositional variables, then $P := [\mathsf{p}]$, $Q := [\mathsf{q}]$, $\neg P := [-\mathsf{p}]$, $P \Rightarrow Q := -[\mathsf{p}]+[\mathsf{q}]$, $P \wedge Q := +[\mathsf{p}]+[\mathsf{q}]$, and $P \vee Q := --[\mathsf{p}]--[\mathsf{q}]$. So, in this example, $\phi := \{P \Rightarrow Q, P\}$ and $\psi := \{R \vee Q\}$.

$$\vdash -\gamma + \psi \begin{cases} \gamma & \{+[\mathsf{q}] \\ \psi & \{- - [\mathsf{r}] - -[\mathsf{q}] \end{cases}$$

In plain English, this example could say what follows, and the interpolant would be the undistributed statement “You are Socrates’ friend”:

$$\vdash -\phi + \psi \begin{cases} \phi & \begin{cases} \text{If you are Plato, you are Socrates' friend.} \\ \text{You are Plato.} \end{cases} \\ \psi & \{\text{You are Greek or you are Socrates' friend.} \end{cases}$$

Example 2 (A syllogistic case). Let $\phi := \{-\mathsf{A} + \mathsf{B}, -\mathsf{C} + \mathsf{A}\}$ and $\psi := \{-\mathsf{C} + \mathsf{B}\}$:

$$\vdash -\phi + \psi \begin{cases} \phi & \begin{cases} -\mathsf{A} + \mathsf{B} \\ -\mathsf{C} + \mathsf{A} \end{cases} \\ \psi & \{-\mathsf{C} + \mathsf{B} \end{cases}$$

Given that $\vdash -\phi + \psi$, there is an interpolant γ such that $unDist(\gamma) \subseteq unDist(\phi) \cap unDist(\psi) = \{\mathsf{B}\}$, and $Dist(\gamma) \subseteq Dist(\phi) \cap Dist(\psi) = \{\mathsf{C}\}$:

$$\vdash -\phi + \gamma \begin{cases} \phi & \begin{cases} -\mathsf{A} + \mathsf{B} \\ -\mathsf{C} + \mathsf{A} \end{cases} \\ \gamma & \{-\mathsf{C} + \mathsf{B} \end{cases}$$

$$\vdash -\gamma + \psi \begin{cases} \gamma & \{-\mathsf{C} + \mathsf{B} \\ \psi & \{-\mathsf{C} + \mathsf{B} \end{cases}$$

This example could say what follows, and the interpolant would be “Every logician$^-$ is mortal$^+$”:

$$\vdash -\phi + \psi \begin{cases} \phi & \begin{cases} \text{Every philosopher is mortal.} \\ \text{Every logician is a philosopher.} \end{cases} \\ \psi & \{\text{Every logician is mortal.} \end{cases}$$

Example 3 (A polisyllogistic case). Let $\phi := \{-\mathsf{A} + \mathsf{B}, -\mathsf{B} + \mathsf{C}, -\mathsf{C} + \mathsf{D}, -\mathsf{D} + \mathsf{E}\}$ and $\psi := \{-\mathsf{A} + \mathsf{E}\}$:

$$\vdash -\phi + \psi \begin{cases} \phi & \begin{cases} -\mathsf{A} + \mathsf{B} \\ -\mathsf{B} + \mathsf{C} \\ -\mathsf{C} + \mathsf{D} \\ -\mathsf{D} + \mathsf{E} \end{cases} \\ \psi & \begin{cases} -\mathsf{A} + \mathsf{E} \end{cases} \end{cases}$$

Given that $\vdash -\phi + \psi$, there is an interpolant γ such that $unDist(\gamma) \subseteq unDist(\phi) \cap unDist(\psi) = \{\mathsf{E}\}$, and $Dist(\gamma) \subseteq Dist(\phi) \cap Dist(\psi) = \{\mathsf{A}\}$:

$$\vdash -\phi + \gamma \begin{cases} \phi & \begin{cases} -\mathsf{A} + \mathsf{B} \\ -\mathsf{B} + \mathsf{C} \\ -\mathsf{C} + \mathsf{D} \\ -\mathsf{D} + \mathsf{E} \end{cases} \\ \gamma & \begin{cases} +(-\mathsf{A} + \mathsf{D}) + (-\mathsf{D} + \mathsf{E}) \end{cases} \end{cases}$$

$$\vdash -\gamma + \psi \begin{cases} \gamma & \begin{cases} +(-\mathsf{A} + \mathsf{D}) + (-\mathsf{D} + \mathsf{E}) \end{cases} \\ \psi & \begin{cases} -\mathsf{A} + \mathsf{E} \end{cases} \end{cases}$$

Following our pattern of examples, this one could say what follows, and an interpolant would be "Every philosopher$^-$ is material$^+$ and every material$^-$ is corruptible$^+$":

$$\vdash -\phi + \psi \begin{cases} \phi & \begin{cases} \text{Every philosopher is human.} \\ \text{Every human is mortal.} \\ \text{Every mortal is material.} \\ \text{Every material is corruptible.} \end{cases} \\ \psi & \begin{cases} \text{Every logician is corruptible.} \end{cases} \end{cases}$$

Example 4 (A relational case). Let $\phi := \{-\mathsf{B} + (+\mathsf{L} + \mathsf{G}), -\mathsf{G} + (+\mathsf{A} + \mathsf{C}), -\mathsf{C} + \mathsf{M}, -(+\mathsf{A} + \mathsf{M}) + \mathsf{F}\}$ and $\psi := \{-\mathsf{B} + (+\mathsf{L} + \mathsf{F})\}$:

$$\vdash -\phi + \psi \begin{cases} \phi & \begin{cases} -\mathsf{B} + (+\mathsf{L} + \mathsf{G}) \\ -\mathsf{G} + (+\mathsf{A} + \mathsf{C}) \\ -\mathsf{C} + \mathsf{M} \\ -(+\mathsf{A} + \mathsf{M}) + \mathsf{F} \end{cases} \\ \psi & \begin{cases} -\mathsf{B} + (+\mathsf{L} + \mathsf{F}) \end{cases} \end{cases}$$

Given that $\vdash -\phi + \psi$, then there is an interpolant γ such that $unDist(\gamma) \subseteq unDist(\phi) \cap unDist(\psi) = \{\mathsf{L}, \mathsf{F}\}$, and $Dist(\gamma) \subseteq Dist(\phi) \cap Dist(\psi) = \{\mathsf{B}\}$:

$$\vdash -\phi + \gamma \begin{cases} \phi & \begin{cases} -\mathsf{B} + (+\mathsf{L} + \mathsf{G}) \\ -\mathsf{G} + (+\mathsf{A} + \mathsf{C}) \\ -\mathsf{C} + \mathsf{M} \\ -(+\mathsf{A} + \mathsf{M}) + \mathsf{F} \end{cases} \\ \gamma & \begin{cases} +(-\mathsf{B} + (+\mathsf{L} + \mathsf{G}))+ \\ (-\mathsf{G} + \mathsf{F}) \end{cases} \end{cases}$$

$$\vdash -\gamma + \psi \begin{cases} \gamma & \begin{cases} +(-\mathsf{B} + (+\mathsf{L} + \mathsf{G}))+ \\ (-\mathsf{G} + \mathsf{F}) \end{cases} \\ \psi & \begin{cases} -\mathsf{B} + (+\mathsf{L} + \mathsf{F}) \end{cases} \end{cases}$$

This example could say what follows, and an interpolant would be "Every logician^{-} likes^{+} some rules^{+} and every rule^{-} is useful^{+}":

$$\vdash -\phi + \psi \begin{cases} \phi & \begin{cases} \text{Every logician likes some rules.} \\ \text{Every rule belongs to some system.} \\ \text{Every system is interesting.} \\ \text{All that belongs to something interesting} \\ \text{is useful.} \end{cases} \\ \psi & \begin{cases} \text{Every logician likes something useful.} \end{cases} \end{cases}$$

Example 5 (Another relational case). Let $\phi := \{+(-\mathsf{A} + (-\mathsf{B})) + \mathsf{C}\}$ and $\psi := \{-(-\mathsf{D} + \mathsf{A}) + (-\mathsf{D} + (-\mathsf{B}))\}$:

$$\vdash -\phi + \psi \begin{cases} \phi & \begin{cases} +(-\mathsf{A} + (-\mathsf{B})) + \mathsf{C} \end{cases} \\ \psi & \begin{cases} -(-\mathsf{D} + \mathsf{A}) + (-\mathsf{D} + (-\mathsf{B})) \end{cases} \end{cases}$$

Since $\vdash -\phi + \psi$, there is an interpolant γ such that $unDist(\gamma) \subseteq unDist(\phi) \cap unDist(\psi) = \varnothing$, and $Dist(\gamma) \subseteq Dist(\phi) \cap Dist(\psi) = \{\mathsf{A}, \mathsf{B}\}$:

$$\vdash -\phi + \gamma \begin{cases} \phi & \begin{cases} +(-\mathsf{A} + (-\mathsf{B})) + \mathsf{C} \end{cases} \\ \gamma & \begin{cases} -\mathsf{A} + (-\mathsf{B}) \end{cases} \end{cases}$$

$$\vdash -\gamma + \psi \begin{cases} \gamma & \{-\mathsf{A} + (-\mathsf{B}) \\ \psi & \{-(-\mathsf{D} + \mathsf{A}) + (-\mathsf{D} + (-\mathsf{B})) \end{cases}$$

Last, this example could say what follows, and an interpolant would be "Every circle$^-$ is unsquared$^-$":

$$\vdash -\phi + \psi \begin{cases} \phi & \{\text{Every circle is unsquared and something.} \\ \psi & \begin{cases}\text{If anyone draws a circle then it draws} \\ \text{something unsquared.}\end{cases} \end{cases}$$

4. *Final Remarks*

Given some superficial links between Lyndon's theorem, term distribution, and Sommers and Englebretsen's logic, in this contribution we have explored some sort of interpolation for TFL. We think this is an interesting result, at least, for the following reason: a research on interpolation for TFL has not been attempted yet, as far as we are aware, and such research suggests TFL has logical properties that *bona fide* logics typically share, which contributes to show that term logics in general, and TFL in particular, far from being superseded, are in the process of revival (Sommers 1982; Englebretsen 1996; Wang 1997; Correia 2017; Simons 2020) and that their death certificate is fake (*contra* Carnap 1930; Russell 1937; Geach 1962).

Finally, having reached our goal, we would like to close this contribution by quickly answering some potential objections.

Objection 1. The notion of interpolation is usually defined in relation to results on proof theory, but this proposal does not develop any proof theory. It is true that interpolation is usually understood in relation to proof theory, but it would be false to claim that our result is void of developments on proof theory, since the main theorem requires the notion of proof in TFL; it is true, however, that we rely on such results or notions without discussion.

Objection 2. The notion of interpolation is usually defined in relation to results on model theory, but this proposal does not develop any model theory. Similarly, it would also be false to claim that our result is unrelated to model theory, since the main theorem requires the notion of distribution, which provides the formal semantics for TFL (Englebretsen 2017).

Objection 3. The notion of interpolation has some clear meaning in classical logic, but there is no clear meaning for this so-called term interpolation. Finally, it would be unsound to argue that there is no clear meaning for term interpolation. Term interpolation would formally explain why distribution has been so important within traditional logic, since it would account for the strength of the *dictum de omni et nullo*, as Hodges (1998) would put it.

REFERENCES

Carnap, Rudolf, 1930, "Die alte und die neue Logik", *Erkenntnis*, vol. 1, pp. 12–26.

Castro-Manzano, J.-Martín, 2020, "Distribution Tableaux, Distribution Models", *Axioms*, vol. 9, no. 2.

Castro-Manzano, J.-Martín, 2018, "A Tableaux Method for Term Logic", *LANMR 2018*, vol. 2264, pp. 1–14.

Castro-Manzano, J.-Martín y Paniel O. Reyes-Cárdenas, 2018, "Term Functor Logic Tableaux", *South American Journal of Logic*, vol. 4, no. 1, pp. 1–22.

Craig, William, 1957, "Linear Reasoning. A New Form of the Herbrand-Gentzen Theorem", *The Journal of Symbolic Logic*, vol. 22, no. 3, pp. 250–268.

Correia, Manuel, 2017, "La lógica aristotélica y sus perspectivas", *Pensamiento*, vol. 73, no. 275, pp. 5–19.

Englebretsen, George, 2017, *Bare Facts and Naked Truths: A New Correspondence Theory of Truth,* Taylor and Francis, Routledge.

Englebretsen, George, 2013, *Robust Reality: An Essay in Formal Ontology*, Philosophische Analyse/Philosophical Analysis, De Gruyter, Berlín.

Englebretsen, George, 1996, *Something to Reckon with: The Logic of Terms*, University of Ottawa Press, Ontario.

Englebretsen, George, 1987, *The New Syllogistic*, Peter Lang, Suiza.

Englebretsen, George and Charles Sayward, 2011, *Philosophical Logic: An Introduction to Advanced Topics*, Bloomsbury Academic, London.

Geach, Peter, 1962, *Reference and Generality: An Examination of Some Medieval and Modern Theories*, Cornell University Press, Ithaca.

Hodges, Wilfrid, 1998, "The Laws of Distribution for Syllogisms", *Notre Dame Journal of Formal Logic*, vol. 39, no. 2, pp. 221–230.

Keynes, John Neville, 1906, *Studies and Exercises in Formal Logic: Including a Generalisation of Logical Processes in Their Application to Complex Inferences*, Macmillan and Company, Nueva York.

Lyndon, Roger Conant, 1959, "An Interpolation Theorem in the Predicate Calculus", *Pacific Journal of Mathematics*, vol. 9, no. 1, pp. 129–142.

Russell, Bertrand, 1937, *A Critical Exposition of the Philosophy of Leibniz: with an Appendix of Leading Passages*, G. Allen and Unwin.

Simons, Peter, 2020, “Term Logic”, *Axioms*, vol. 9, no. 18.
Sommers, Fred, 1975, “Distribution Matters”, *Mind*, vol. 84, no. 1, pp. 27–46.
Sommers, Fred, 1982, *The Logic of Natural Language*, Oxford University Press, Oxford.
Sommers, Fred, 1989, “Predication in the Logic Terms”, *Notre Dame Journal of Formal Logic*, vol. 31, no. 1, pp. 106–126.
Sommers, Fred and George Englebretsen, 2000, *An Invitation to Formal Reasoning: The Logic of Terms*, Ashgate, Farnham, UK.
Wang, Pei, 1997, *Return to Term Logic*, Working Notes of the IJCAI Workshop on Abduction and Induction in AI.
Wilson, Fred, 1987, “The Distribution of Terms: A Defense of the Traditional Doctrine”, *Notre Dame Journal of Formal Logic*, vol. 28, no. 3, pp. 439–454.

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