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LOGICAL OMNISCIENCE AS A CONDITIONALITY ISSUE. A MULTI-MODAL APPROACH

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SUMMARY: Many solutions to the problem of Logical Omniscience assume that this arises from the behavior of the epistemic operators. However, few proposals have criticized the assumption that material implication accurately accounts for conditionality. This paper aims to show how Multi-Modal Logic can be used to criticize this assumption. After reviewing a kind of fusion semantics that incorporates a set of epistemic states to the models, serious systems of Multi-Modal Logic are used to criticize both the validity of Closure Principles for Knowledge and Belief and a version of Logical Omniscience that uses the strict conditional. The machinery is also modified to explore Logical Omniscience in logics based on Conditional Logic, Intuitionistic Logic, and a pair of weak Relevant Logics.

KEYWORDS: conditionality, closure principle, fusion semantics, multi-modal logic, non-normal modal logic

RESUMEN: Muchas soluciones al problema de la omnisciencia lógica critican el comportamiento de los operadores epistémicos. Sin embargo, pocos han criticado el supuesto de que la implicación material representa adecuadamente la condicionalidad. Este artículo propone usar lógicas multimodales para criticar este supuesto. Después de revisar un tipo de semántica de fusión que incorpora estados epistémicos a los modelos, se usan varios sistemas de lógica multimodal para criticar tanto la validez de la clausura para el conocimiento y la creencia como también una versión de la omnisciencia lógica que usa el condicional estricto. La maquinaria también se modifica para explorar la omnisciencia lógica en lógicas basadas en la lógica condicional, la lógica intuicionista y un par de lógicas relevantes débiles.

PALABRAS CLAVE: condicionalidad, clausura epistémica, semántica de fusión, lógica multimodal, lógica modal no normal

Introduction

One of the mainstream problems of Epistemic/Doxastic Logic is that they fall into Logical Omniscience, that is, an agent knows all the logical consequences of any given set of propositions. Many solutions to the problem assume that this arises from the behavior of the epistemic operators. However, few proposals have criticized the assumption that material implication is at the base (the research on Intuitionistic-Epistemic Logic is a remarkable exception). This paper aims to show how Multi-Modal Logic can be used to criticize Logical Omniscience using different notions of conditionality.[1]

The structure of the paper will be the following. First, as an illustration of a Multi-Modal logic, I will develop the axiomatic system for $T_{\Box}/T_K/D_B^*$, and I will make some comments on its fusion semantics that adopt a set, E, of epistemic states. I will also point out the necessary modifications if we use a system based on $S5$ for two logics. After that, I will develop a critique of the Closure Principle for serious modal operators. We will look at Logical Omniscience with material implication, how Multi-Modal Logic gives a new version, and how non-normal semantics may give two possible solutions to Strict Logical Omniscience. Then I will philosophically discuss the machinery. In the last section, I will give further examples of multimodal systems that give logical principles a new framework. Properly, I will develop some remarkable features of Epistemic Logics based on Conditional Logic, Intuitionistic Logic, and some weak Relevant Logics.

1. *Some Multi-Modal Logics*

For the sake of brevity, I will take for granted that the reader is familiar with Alethic, Epistemic, Doxastic, and Tense Logic. Nevertheless,

[1] Previous versions of this paper were developed in some lectures, namely, at the 4th French-Mexican Seminar of Advance Topics in Philosophy of Science and Mathematics in 2022, the Seminar of Research Developments in Universidad Autónoma Metropolitana in 2022, and the 56th Colloquium of the Sociedad Matemática Mexicana in 2023. Most of the statements made in this paper belong to my master's degree thesis (Sánchez-Hernández 2022a), where I also developed tableaux systems that are sound and complete according to their semantics. Indeed, while most of the statements made in this paper can be examined with that machinery, there are two exceptions. The first is the case of the systems that use sphere semantics for counterfactual conditionals in sec. 4.1. The second is that of the Relevant Epistemic Logics in sec. 4.3, but I give their tableaux systems in Appendix 1. See also Sánchez-Hernández (2022b), a thesis preview in which some results were published. In particular, sec. 2 of this article is a revision of sec. 5 of that article. However, in my thesis, the philosophical discussion was subordinate to the machinery. In this paper, I try to articulate a discussion that can benefit further investigations. Appendix 2 offers briefly an application of non-classical logics to the study of Deontic Logic, especially in discussing Hume's critique of is-ought inferences and Moore's Naturalistic Fallacy proposal. Finally, I would like to thank two anonymous referees for their insights, which improved this paper.

there is an assumption I think is worth questioning to develop Multi-Modal Logics. When we switch from Alethic Logic to Tense Logic, it is usual to modify slightly the Kripke models. $\langle W, R, \nu\rangle$ changes to $\langle T, <, \nu\rangle$. However, this change does not happen when we switch from Alethic Logic to Epistemic Logic, the assumption is that possible worlds work too. I think this practice is wrong. I suggest that, in Epistemic and Doxastic Logics, we should also slightly modify Kripke models appealing to a set of epistemic states, E, so a model for an Epistemic-Doxastic Logic should be $\langle E, \Psi^K, \Psi^B, \nu\rangle$, where Ψ^K is the accessibility relation for K, and Ψ^B that for B. Certainly, this remark is more philosophical than technical. We can get similar results by ignoring it, as we shall see. Indeed, if we work with a single Modal Logic, this consideration is completely irrelevant from a technical point of view. However, the philosophical discussion has a new path to follow if we make the change.

Let us begin with the syntax of the language. Let M and $\widehat{M}$ be a modal operator and its dual, say, K and $\widehat{K}$. Let $\mathscr{P} = \{p : p \text{ is a propositional parameter}\}$. In all of the languages, the grammar is the same: if $p \in \mathscr{P}$, then p is a (well-formed) formula —φ—; if φ and ψ are formulas, so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$, $M\varphi$, and $\widehat{M}\varphi$. Sometimes I will use the following definitions: for contingency, $\nabla p =_{def} \Diamond p \wedge \Diamond\neg p$; for "Always" in Tense Logic, $Ap =_{def} Hp \wedge p \wedge Gp$; for its dual —"Sometimes"—, $Sp =_{def} \neg A\neg p \leftrightarrow (Pp \vee p \vee Fp)$. As usual, $p \dashv q =_{def} \Box(p \rightarrow q)$. Since we will be working with serious modal systems and their combinations, I will use S_M to denote the system S for the modal operator M, *e.g.*, $S4_K$ is $S4$ as an Epistemic Logic.

Take an alethic-epistemic-doxastic language, $\mathscr{L}_{\Box KB} = \mathscr{L}_{\Box} \cup \mathscr{L}_K \cup \mathscr{L}_B$. As an illustration of an Alethic-Epistemic-Doxastic Logic, I will develop the logic $T_{\Box}/T_K/D^*_B$ (the $*$ is due to the adoption of (K/B) axiom below). Its axiomatic system is the following:[2]

(PC) All the tautologies, $\vdash \varphi$, of classical propositional calculus

(US) If $\vdash \varphi$, p is part of φ, and φ' is the same as φ except that ψ uniformly substitutes p, $[p/\psi]$, then $\vdash \varphi'$

[2] Notice that (K$\Box$) refers to K axiom with the operator $\Box$, while (KK) refers to its epistemic version. Nevertheless, despite the good intentions of clarity, the number of elements may be quite confusing. Hence, by recommendation of an anonymous referee, the reader may find useful Appendix 3 as summary of all the abreviations of the logical principles reviewed in this paper.

(MP) If $\vdash \varphi \to \psi$ and $\vdash \varphi$, then $\vdash \psi$

(N$\Box$) If $\vdash \varphi$, then $\vdash \Box\varphi$

(NK) If $\vdash \varphi$, then $\vdash K\varphi$

(NB) If $\vdash \varphi$, then $\vdash B\varphi$

(K$\Box$) $\Box(p \to q) \to (\Box p \to \Box q)$

(KK) $K(p \to q) \to (Kp \to Kq)$

(KB) $B(p \to q) \to (Bp \to Bq)$

(T$\Box$) $\Box p \to p$

(TK) $Kp \to p$

(DB) $Bp \to \neg B \neg p$

(K/B) $Kp \to Bp$

Now, let us turn to the fusion semantics for its language (for an overview of this kind of semantics, see Smets and Velázquez-Quesada (2023, sec. 3.1)). An interpretation, $\mathfrak{I}$, is a structure $\langle W, E, \{R^{\Box}_e : e \in E\}, \{\Psi^K_w : w \in W\}, \{\Psi^B_w : w \in W\}, \nu\rangle$. W and E are non-empty sets of possible worlds and epistemic states respectively. The accessibility relations for modal, epistemic, and doxastic operators are relativized properly, *e.g.*, the accessibility relation for $\Box$, $R^{\Box}$, is relative to epistemic states, $\{R^{\Box}_e : e \in E\}$. Finally, ν is a map from $W \times E \times \mathscr{P}$ to $\{0, 1\}$. Hence, $\nu_{w/e}(p) = 1$ should be read as "at the world w in the epistemic state e, p is true".

The truth conditions are those for modal, epistemic, and doxastic logics, except that they are relativized properly. $\vee$, $\to$, $\leftrightarrow$, and $\dashv$ are defined as usual. Here we have the truth conditions for $\neg$, $\wedge$, $\Box$, K, and B:

$\nu_{w/e}(\neg\varphi) = 1$	iff	$\nu_{w/e}(\varphi) = 0$
$\nu_{w/e}(\varphi \wedge \psi) = 1$	iff	$\nu_{w/e}(\varphi) = \nu_{w/e}(\psi) = 1$
$\nu_{w/e}(\Box\varphi) = 1$	iff	for all $w' \in W$ such that $wR^{\Box}_e w'$, $\nu_{w'/e}(\varphi) = 1$
$\nu_{w/e}(K\varphi) = 1$	iff	for all $e' \in E$ such that $e\Psi^K_w e'$, $\nu_{w/e'}(\varphi) = 1$
$\nu_{w/e}(B\varphi) = 1$	iff	for all $e' \in E$ such that $e\Psi^B_w e'$, $\nu_{w/e'}(\varphi) = 1$

The truth conditions for $\Diamond$, $\widehat{K}$, and $\widehat{B}$ are similar, they only change "all" for "some". Since truth conditions are only relativized, then $M\varphi \leftrightarrow \neg\widehat{M}\neg\varphi$ for all the operators in the language.

To have all the axioms for $T_{\Box}/T_K/D^*_B$, we need to add some constraints to the semantics. To have (T$\Box$), (TK), (DB), and (K/B), we add the following constraints respectively:

For all $w \in W$ and $e \in E$, $wR^{\Box}_e w$
For all $w \in W$ and $e \in E$, $e\Psi^K_w e$
For all $w \in W$ and $e \in E$, there is a $e' \in E$ such that $e\Psi^B_w e'$
For all $w \in W$, $\{\Psi^B_w : w \in W\} \subseteq \{\Psi^K_w : w \in W\}$

Semantic validity is defined by means of truth preservation over worlds and epistemic states. Let Φ be any set of well-formed formulas. Then:

$\Phi \models \psi$ iff for all $\mathfrak{I}$ and for all $w \in W$ and for all $e \in E$, if, for all $\varphi \in \Phi$, $\nu_{w/e}(\varphi) = 1$, then $\nu_{w/e}(\psi) = 1$

Since truth conditions are only relativized, these do not affect the original validity of the elements of the axiomatic system. A $T_{\Box}/T_K/D^*_B$-interpretation where $E = \{e\}$ is the same as a $T_{\Box}$-interpretation and a $T_{\Box}/T_K/D^*_B$-interpretation where $W = \{w\}$ is the same as a T_K/D^*_B-interpretation. Hence, the axiomatic system is sound according to the semantics.

$T_{\Box}/T_K/D^*_B$ language is more expressive than any of its components. This language can be used to explore both our epistemic and doxastic attitudes towards alethic logic, *e.g.*, $K\Box p \dashv \neg B\nabla p$, and the alethic modality of our knowledge and beliefs, *e.g.*, $(Kp \wedge \Diamond\neg p) \dashv \nabla Kp$ —this is a theorem due to von Wright (1985, p. 68). It is easy to show that the last two formulas are provable in $T_{\Box}/T_K/D^*_B$.

The $T_{\Box}/T_K/D^*_B$ fusion semantics have some notable features. Let us see some reflections on the adoption of the set E.

First, they were developed as an analogy to the semantics for the modal-tense system $K_{\Box}/K_{GH}$ (named MT in Sánchez-Hernández 2022c). The semantics for $\mathscr{L}_{\Box GH}$ is a structure $\langle W, T, \{R_t : t \in T\}, \{<_w : w \in W\}, \nu\rangle$. W and T are non-empty sets of possible worlds and times respectively. $\{R_t : t \in T\}$ indicates that accessibility relations change over time. This may be represented as the fact that possibilities change over time, *e.g.*, "In 1932 it was possible for Great Britain to avoid war with Germany; but in 1937 it was impossible"

(Thomason 1984, p. 207). $\{<_w: w \in W\}$ would represent that possible worlds differ in their time orderings. There may be a world where time does not have an end, $AF\top$, although there may be an unfortunate world where time does end, $AF\bot$ (Correia and Rosenkranz 2020, pp. 6–7). However, for the sake of simplicity, most logicians prefer an absolute order for the worlds, $<$, that is, all of them share the same order (Thomason 1984, p. 208). Let us call the logic $K_\Box/K^*_{GH}$ if we have $<$ instead of $\{<_w: w \in W\}$. ν is the same as for $T_\Box/T_K/D^*_B$ *mutatis mutandis*. By analogy, in the $T_\Box/T_K/D^*_B$ fusion semantics, $\{R^\Box_e : e \in E\}$ along with ν would mean that possible worlds semantics for alethic operators are relative to an agent's epistemic states. Hence, possible worlds only make sense inside an agent's mind. The machinery seems to force us to have an actualist interpretation of possible world semantics for Modal Logic. Furthermore, if we were to develop a Multiagent Epistemic Logic, they may differ on what they know/believe about modal sentences, $K_a\Box p$, $B_b\neg\Box p$, $B_c(\Diamond q \rightarrow \Box p)$. On the other hand, $\{\Psi^K_w : w \in W\}$ and $\{\Psi^B_w : w \in W\}$ along with ν would mean that our epistemic states settings change along different states of affairs, which is no extraordinary statement given that we change our beliefs and knowledge along time, and so along circumstances.

Now, these fusion semantics assume a difference between the members of W and those of E —even if it is not easy to say *prima facie* where the distinction lies. However, to develop a Multi-Modal Logic such a distinction is not necessary. Over the last decade, Rönnedal (2012a, 2012b, 2015, 2018, 2021) has developed several multimodal systems, the most ambitious of which is a Quantified Temporal Alethic Boulesic Doxastic Logic —a Boulesic Logic deals with notions concerning our will, such as "x wants p to be the case". Nevertheless, none of these systems requires that alethic operators and epistemic/doxastic/boulesic operators are semantically based on different sets for their accessibility relations, they are all based on possible worlds, W, but tense operators do have their base on a set of times, T. Hence, there is no technical need to embrace the set E.

However, I think it is philosophically right to adopt the set E. Besides the possible actualist interpretation of Modal Logic, consider the following regarding the semantics for Deontic Logic. Let us say that a modal operator's *realm* is the set on which we base its accessibility relation. Thus, the realm for tense operators is T, and that for alethic operators is W. In this framework, it seems plausible that the epistemic/doxastic operators' realm is E. We could even say that it is plausible that boulesic operators —want, deny, wish— are proper of the realm of E. Nevertheless, if we were to develop a Deontic Logic, with the operators for Obligation —O— and Permission —P—, the question of their realm is less clear. Certainly, it is not T (although the temporalization of Deontic Logic is desirable since laws and rights change over time). However, if its realm is W, moral necessity would be in the same realm as alethic operators. Does moral necessity work at the same level as reality? Can we accept such commitment to the reality/objectivity of moral obligations? This is doubtful: although moral obligations usually do work to keep a good society, there may be circumstances where it is acceptable to go against them (Rachels 2003, ch. 9). To state that E is the realm for O and P is not *prima facie* a better option. Since the T axiom fails for moral necessity, it seems moral necessity would be similar to belief. Thus, moral obligations may be a mere subjectivity issue (on subjectivism, see Rachels (2003, ch. 3). Nevertheless, since Deontic Logic cannot be impersonal for moral duties are always about the agents (a person who is a parent —president, guard, etc.— has more obligations than someone who is not), Deontic Logic resembles more to Epistemic-Doxastic Logic than it does to Alethic-Tense Logic. If we are to adopt a Deontic Logic, I think the question for its realm should be properly answered; especially if we would like to address the problem of the inference from "is" to "ought" (for an overview in the context of Evolutionary Ethics, see Rachels 1990 (pp. 66–70) and Thompson 2022 (sec. 5); I review the problem of is-ought inferences in the Appendix 2 of this paper with a Multi-Modal Perspective). Hence, adopting the set E has serious philosophical implications. Thus, I think the $T_\Box/T_K/D^*_B$ fusion semantics is worth philosophical consideration.[3]

Let us turn back to the formal issues. $T_\Box/T_K/D^*_B$ is my favorite normal system for I think it is the minimum we can ask for an Alethic-Epistemic-Doxastic Normal Logic. However, we can get stronger logics by adding other axioms, and doing the proper relativizations as before:

(4□)	$\Box p \to \Box\Box p$
(5□)	$\Diamond p \to \Box\Diamond p$
(4K)	$Kp \to KKp$
(5K)	$\neg Kp \to K\neg Kp$

[3] It is worth noting that it is possible to temporalize $T_\Box/T_K/D^*_B$ adding the set T to the semantics and doing the proper relativization. See Sánchez-Hernández 2022a (ch. 5).

(4B) $Bp \rightarrow BBp$
(5B) $\neg Bp \rightarrow B\neg Bp$
(B/BK) $Bp \rightarrow BKp$
(B/KB) $Bp \rightarrow KBp$

Notice that (K/B), (B/BK), and (B/KB) are interactions between Knowledge and Belief. The principles —whether they are axioms or theorems— that indicate such interactions are commonly called *bridge principles* —in First-Order Modal Logic, the Barcan Formulas are considered bridge principles insofar they mingle quantification and modality. Nevertheless, notice we do not have axioms that work as bridge principles for Knowledge/Belief and Alethic Modality —although it is easy to get $Kp \rightarrow \Diamond p$ in $T_\Box/T_K$—. These do exist. Everything about further axioms is straightforward until we reach the cases where $S5$ is the base for two logics. Let $\sim_x^M$ be a universal relation for the M operator relativized to x. Let us develop the semantics for $S5_\Box/S5_K$. A model would be a structure $\langle W, E, \{\sim_e^\Box : e \in E\}, \{\sim_w^K : w \in W\}, \nu\rangle$. The truth conditions for $\Box$ and K are now the following:

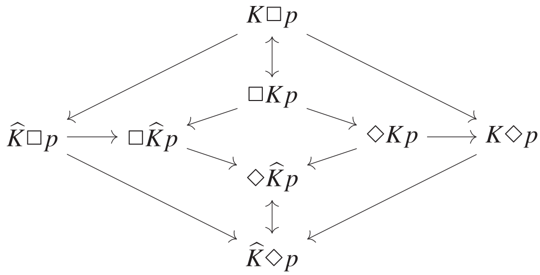
$$\nu_{w/e}(\Box\varphi) = 1 \text{ iff, for all } w' \in W, \nu_{w'/e}(\varphi) = 1$$

$$\nu_{w/e}(K\varphi) = 1 \text{ iff, for all } e' \in E, \nu_{w/e'}(\varphi) = 1$$

Beside the obvious additions to the axiomatic system, we add two more axioms:

(ABF) $K\Box p \rightarrow \Box Kp$
(ABCF) $\Box Kp \rightarrow K\Box p$

Their names are because they are analogous to the Barcan Formula (BF) and the Converse Barcan Formula (CBF) in First-Order Modal Logic. Indeed, we can modify the lattice that Kneale and Kneale (1962, p. 614) developed to those formulas to obtain the following lattice (the outer diamond has sentences about the epistemic attitudes towards modal logic while the inner diamond has sentences about the alethic conditions of epistemic operators):



Nevertheless, (ABF) and (ABCF) are not free of philosophical problems, but we will turn back to these in the next section.

It is worth noting that the temporal-alethic versions of these formulas, $A\Box p \rightarrow \Box Ap$ and $\Box Ap \rightarrow A\Box p$, require at least $S5_{\Box}/K^{*}_{GH}$, that is, we need $<$ instead of $\{<_w \colon w \in W\}$. The same applies *mutatis mutandis* to temporal-epistemic versions, $AKp \rightarrow KAp$ and $KAp \rightarrow AKp$.

2. *On the Strict Closure Principle*

Like in Modal Logic, some principles are valid with material implication, $\rightarrow$, which are not valid with strict implication, $\strictif$. The same happens with Multi-Modal Logics. In this section, Multi-Modal analysis will be used to show that the Closure Principle for Knowledge fails with strict implication in weaker logics than $S5_{\Box}/S5_K$ since there may be circumstances where our knowledge fails. Indeed, I will show that Multi-Modal Logics may give a revindication of Dretske's penetrating operators notion on this issue.

The closure principle for $\Box$ —(K$\Box$)— is valid with $\strictif$ for any logic based on $K_{\Box}$.

(K$\Box$+) $\Box(p \strictif q) \strictif (\Box p \strictif \Box q)$

Insofar $p \strictif q =_{def} \Box(p \rightarrow q)$, a variation is possible.

(K$\Box\pm$) $(p \strictif q) \strictif (\Box p \strictif \Box q)$

This is valid if we add (4$\Box$) to the equation.

However, to have the Strict Closure Principle for Knowledge

(KK+) $K(p \strictif q) \strictif (Kp \strictif Kq)$

we need both bases to be $S5$, that is, $S5_{\Box}/S5_{K}$.

However, since $S5$ is not an accurate system for doxastic modality —due to the failure of the T axiom—, even if the base alethic logic is $S5_{\Box}$, we cannot have the strict closure for belief:

(KB+) $B(p \strictif q) \strictif (Bp \strictif Bq)$

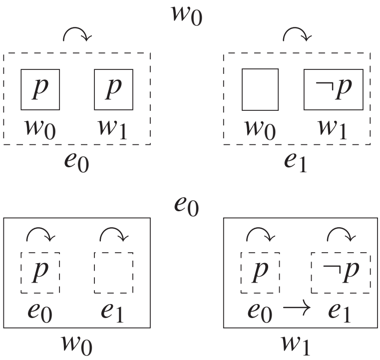
The reason why (KK+) can be valid if one uses $S5_{\Box}/S5_{K}$ is because of the validity of (ABF). Here is an axiomatic proof:

(I)	$K(p \to q) \to (Kp \to Kq)$	Take (KK)
(II)	$\Box(K(p \to q) \to (Kp \to Kq))$	By (N$\Box$) to (I)
(III)	$\Box K(p \to q) \to \Box(Kp \to Kq)$	By (K$\Box$) to (II)
(IV)	$K\Box(p \to q) \to \Box K(p \to q)$	(US), $[p/p \to q]$, to (ABF)
(V)	$K\Box(p \to q) \to \Box(Kp \to Kq)$	Hyp. Syl. by (III) and (IV)
(VI)	$\Box(K\Box(p \to q) \to \Box(Kp \to Kq))$	By (N$\Box$) to (V)
(VII)	$K(p \strictif q) \strictif (Kp \strictif Kq)$	Definition of $\strictif$ to (VI)

Now, in alethic-tense logic literature, it is usual to distinguish between *absolute necessity*, when something is always necessary, $A\Box\varphi$, and *historical necessity*, when something is necessary at some particular time, that is, $\Box\varphi$, $P\Box\varphi$, $F\Box\varphi$, or $S\Box\varphi$ (See Thomason 1984, sec. 2) for an introduction to historical necessity, Rönnedahl 2021 (sec. 1 and 2) for literature and a semantic approach, and Sánchez-Hernández 2022c (sec. 3) for a discussion on the interaction between $A\Box\varphi$ and $\Box\varphi$). By analogy, in alethic-epistemic frameworks, we should also distinguish between *absolute knowledge* (pace Hegel), when an agent knows something under all circumstances, $\Box K\varphi$, and *relative knowledge*, that is, $K\varphi$, $\Diamond K\varphi$, or $\nabla K\varphi$.

Now, although (ABF) and (ABCF) are valid in $S5_{\Box}/S5_{K}$, one might doubt their plausibility.

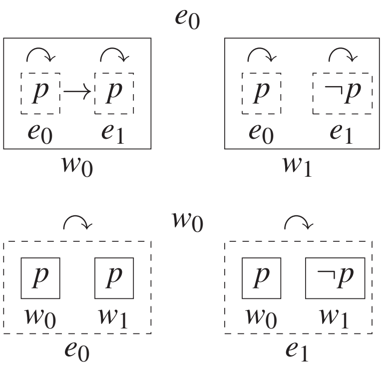
(ABF) fails because even if we know something is necessary, $K\Box p$, there may be a state of affairs where we do not know whether it is true, $\Diamond(\neg Kp \wedge \neg K\neg p)$. Here is a model of $S5_{\Box}/S4_{K}$ to that effect (worlds are depicted by complete boxes while epistemic states with dashed boxes; there is no need to indicate the alethic relations for they are universal; the model above illustrates an epistemic situation while the model below exemplifies an alethic situation; the values of the parameters are the same in both models (*e.g.*, $\nu_{w1/e_1}(\neg p) = 1$), however the relations change over worlds and epistemic states).



This may be disappointing for a philosopher who has finally found a necessary truth —the holy grail of her theory—, there may be circumstances where she never thought about such proposition. Let p be *water* = H_2O. Even if $K\Box p$, there may be a world where chemistry has not made such progress to state that *water* = H_2O, therefore $\neg\Box Kp$.[4]

(ABCF) fails if we think *à la Hume*. Even if we had absolute knowledge about something, $\Box Kp$, we could not conclude that we know it as necessary, $\neg K\Box p$, if we can conceive that it is possibly false, $\widehat{K}\Diamond\neg p$. Here is a model of $S5_{\Box}/S4_K$ to that effect.

[4] The reader may want to point out that, in First-Order Alethic Epistemic Logic, $\forall x\forall y(x = y \to Kx = y)$ holds; even more, it is provable that *water* = $H_2O \to \Box K$ *water* = H_2O. One advantage of having a Multi-Modal Logic is that a logician may accept a principle for one kind of modality but not for another (consider the T axiom). However, in the case of an Alethic-Epistemic First-Order Logic, I have not found the way to develop the semantics to have $\forall x\forall y(x = y \to \Box x = y)$ without $\forall x\forall y(x = y \to Kx = y)$. Certainly, the first seems valid if it means that an object is necessarily identical to itself —granted we use rigid designators—, but the second seems less plausible since many things are identical that we do not know that they are so (Priest 2008, sec. 17.3). The development of such a system is something left on the agenda. Nevertheless, for a system with both principles, see Sánchez-Hernández 2022a (ch. 10).



The issue is that Hume thought about this over time to deny the induction of future events. Even if you always knew that the sun rises from the east, AKp, you could not know that it will always do, $\neg KAp$, if you can conceive that it will not at some future time, $\widehat{K}F\neg p$.

Now, there is a well-known critique of the Closure Principle due to Dretske (1970). According to him, there are some operators, M, such that "if p entails q, then Mp entails Mq",[5] he qualifies them as "*penetrating operator[s]*" (1970, p. 1007). The concept comes in degrees, for he also accepted that there are non-penetrating and semi-penetrating operators. Indeed, he believed that "all epistemic operators are semi-penetrating" (1970, p. 1009). To support his thesis, he states:

> When we are dealing with epistemic operators, it becomes crucial to specify whether the agent in question knows that p entails q. That is to say, p may entail q, s may know that p, but he may not know that q *because*, and perhaps *only* because, he fails to appreciate the fact that p entails q. When q is a simple logical consequence of p we do not expect this to happen, but when the propositions become very complex, or the relationship between them [is] very complex, this might occur. Let p be a set of axioms, q a theorem. s's knowing p does not entail s's knowing q just because p entails q. (p. 1010)

Now, Multi-Modal Logics can give a new perspective on Dretske's ideas. Assume $\dashv$ has entailment force. Then $\Box$ is a penetrating operator in any logic as strong as $K4_{\Box}$ for $(p \dashv q) \dashv (\Box p \dashv \Box q)$ is valid.

[5] Dretske's notation has been changed to keep uniformity.

However, even if an agent knows that $p \prec q$ —which is a stronger assumption than that of Dretske—, $K(p \prec q)$, that does not entail that knowing p entails knowing q, $Kp \prec Kq$. The problem with logics weaker than $S5_{\Box}/S5_K$ is their lack of $K\Box p \to \Box Kp$. However, $K\Box p \to \Box Kp$ and $\Box Kp \to K\Box p$ are quite dubious principles as we have seen. Particularly, the failure of $K\Box p \to \Box Kp$ suggests that the failure of $K(p \prec q) \prec (Kp \prec Kq)$ is because our knowledge can not be absolute even if it is originally about a necessary proposition. There are some states of affairs where our knowledge fails. K is not as penetrative as $\Box$ is.

3 . *Logical Omniscience and Non-Normal Multi-Modal Logics*

One of the mainstream problems of Epistemic Logic is Logical Omniscience, the idea that an agent knows all the logical implications of any given set of propositions. In this section, I will show that, despite the failure of the Closure Principle for Knowledge with strict implication, Logical Omniscience is attainable in Normal Multi-Modal Logics. As a quick solution, I will show how adding non-normal worlds and epistemic states may solve the problem. A remarkable feature of this technical approach is that Non-Normal Modal Logic and Normal Modal Logic can be combined. I will finish this section with a discussion on what non-normal worlds and non-normal epistemic states may be.

There are many ways to state Logical Omniscience.[6] However, a standard form is the following for the material conditional:

(MLO) If $\vdash \varphi \to \psi$, then $\vdash K\varphi \to K\psi$

Indeed, notice that according to Dretske's definition of penetrating operators, Logical Omniscience is an instance of a broader issue: for any operator, M, we can ask whether $\vdash \varphi \to \psi$ entails $\vdash M\varphi \to M\psi$.

Given fusion semantics for Multi-Modal Logics, it is natural to wonder whether we can have a new version of Logical Omniscience, namely, a strict one.

(SLO) If $\vdash \varphi \prec \psi$, then $\vdash K\varphi \prec K\psi$

This is plausible if one assumes that $\prec$ has entailment force.

In any logic based on $T_{\Box}$, we have the strict version of *modus ponens*:

(SMP) If $\vdash \varphi \prec \psi$ and $\vdash \varphi$, $\vdash \psi$

[6] See Meyer 2003 (sec. 4.1) for a list.

In $S5_{\Box}/S5_K$, (SLO) clearly holds:

Suppose	$\vdash \varphi \strictif \psi$
By (NK),	$\vdash K(\varphi \strictif \psi)$
By (KK+) and (SMP),	$\vdash K\varphi \strictif K\psi$
Therefore,	If $\vdash \varphi \strictif \psi$, then $\vdash K\varphi \strictif K\psi$

Given the failure of (KK+) in any system weaker than $S5_{\Box}/S5_K$, arguably it should also fail (SLO). However, even in $K_{\Box}/K_K$ it is valid that:

(QKK+) $\Box K(p \to q) \strictif (Kp \strictif Kq)$

(Q stands for "quasi", so here we have a Quasiclosure Principle for Knowledge. We can get this principle by applying (N□) to line III of the proof for (KK+).) Notice that the converse of (N□), (CN□), is valid: if $\vdash \Box\varphi$, then $\vdash \varphi$ (this is not the T axiom). Hence, any valid strict implication, $\vdash \varphi \strictif \psi$, is a valid material implication, $\vdash \varphi \to \psi$. The proof for (SLO) in $T_{\Box}/K_K$ —and, *a fortiori* in stronger systems— is the following:

Suppose	$\vdash \varphi \strictif \psi$
By (CN□),	$\vdash \varphi \to \psi$
By (NK),	$\vdash K(\varphi \to \psi)$
By (N□),	$\vdash \Box K(\varphi \to \psi)$
By (QKK+) and (SMP),	$\vdash K\varphi \strictif K\psi$
Therefore,	If $\vdash \varphi \strictif \psi$, then $\vdash K\varphi \strictif K\psi$

Now, (NK) and (KK) entail (MLO). Because of the former, the agent knows all classical tautologies. The latter allows her to close her knowledge under material implication. (SLO) only adds (N□) to the equation. Since it does not matter whether the Closure Principle is material —(KK)— or strict —(KK+)—, perhaps the most straightforward way to avoid (MLO) and (SLO) is to invalidate (NK) by adding non-normal worlds to the semantics. Let W_N and E_N be subsets of W and E respectively, namely, those of normal worlds and normal epistemic states. A model for a non-normal single modal logic would be $\langle W, W_N, R^{\Box}, \nu\rangle$. W and $R^{\Box}$ are as usual. If $w \in W_N$, the truth conditions for □ and ◊ are as usual. In non-normal worlds, $w \in W - W_N$, the values of modal formulas do not depend on φ. Now their truth conditions are stated as follows:

$$\nu_w(\Box\varphi) = 0$$
$$\nu_w(\Diamond\varphi) = 1$$

Validity would be defined in terms of truth preservation over normal worlds:

$$\Phi \models \psi \quad \text{iff} \quad \text{for all } \langle W, W_N, R^{\Box}, \nu\rangle \text{ and for all } w \in W_N, \text{ if, for all } \varphi \in \Phi,\ \nu_w(\varphi) = 1, \text{ then } \nu_w(\psi) = 1$$

In this framework, $\models p \vee \neg p$ entails $\models \Box(p \vee \neg p)$. However, $\models \Box\Box(p \vee \neg p)$ fails since no formula of the form $\Box\varphi$ holds in non-normal worlds (Priest 2008, sec. 4.4). This strategy can be applied *mutatis mutandis* to Epistemic Logic. The same strategy can be applied to Multi-Modal Logics to base them on Non-Normal Modal Logics. Now an interpretation, $\mathfrak{I}$, is $\langle W, W_N, E, E_N, \{R^{\Box}_e : e \in E\}, \{\Psi^K_w : w \in W\}, \{\Psi^B_w : w \in W\}, \nu\rangle$. Validity is defined in terms of truth preservation over normal worlds and epistemic states:

$$\Phi \models \psi \quad \text{iff} \quad \text{for all } \mathfrak{I} \text{ for all } w \in W_N \text{ and for all } e \in E_N, \text{ if, for all } \varphi \in \Phi,\ \nu_{w/e}(\varphi) = 1, \text{ then } \nu_{w/e}(\psi) = 1$$

Of course, it is possible to have only normal possible worlds —so $W = W_N$— or only normal epistemic states —so $E = E_N$. Indeed, it is worth noting that fusion semantics were originally developed to combine Normal Modal Logics; however, the combination of Normal and Non-Normal Modal Logics was left on the agenda (Carnielli and Coniglio 2020, sec. 4.1). What I have shown provides an easy strategy to combine logics of both kinds. There may be logics that do not admit non-normal versions, say, Tense Logic. However, it is possible to have a Non-Normal Alethic Logic combined with a Normal Tense Logic. The philosophical discussion below may shed some light on this matter.

With the machinery above described, we are in a position to choose. On the one hand, if we add non-normal worlds to the semantics, (N$\Box$) is invalid, and so is (SLO); however, (MLO) is still valid. On the other hand, if we add non-normal epistemic states to the semantics, (NK) is invalid, and so are (SLO) and (MLO). Hence, we have two options to make (SLO) fail.

The reader may want to point out that with non-normal worlds in the semantics (K$\Box\pm$) is invalid, so $\Box$ is a non-penetrating operator as Dretske describes them. This does not have to be the case. If $\{R^{\Box}_e : e \in E\}$ is constrained to be relatively transitive, then (K$\Box\pm$) is valid while (KK+) and (SLO) are invalid. Hence, $\Box$ is a penetrating operator while K is not. Nevertheless, if we use the system $S2_{\Box}/S2_K$ —being $S2$ the non-normal version of T, see Priest

(2008, sec. 4.2)—, then the agent can make use of strict *modus ponens*:

(*K*SMP) $(K(p \dashv q) \wedge Kp) \dashv Kq$

Notice that (*K*SMP) and (K*K*+) are equivalent when we use $\rightarrow$. Hence, in Multi-Modal Logic the equivalence usually fails.

Now, although the introduction of non-normal worlds and epistemic states gives technical results, it is not easy to say exactly what we are embracing —if it was not already difficult to differentiate the members of W from those of E. In the remainder of the section, I want to explore this issue.

Let us begin with non-normal worlds. Berto and Jago (2023) —although see (2019) for a wider development— suggest that impossible worlds are mainly developed in four ways:

(1) *Impossible Ways*: Just like possible worlds are worlds where possible things happen, impossible worlds are often characterized as ways things could not have been.

(2) *Logical Violators*: Impossible worlds are worlds where the laws of logic fail.

(3) *Classical Logic Violators*: Impossible worlds are worlds where the laws of classical logic fail. A world, w, where Intuitionistic Logic is accurate is an impossible world.

(4) *Contradiction-Realizers*: an impossible world is a world where sentences of the form $\varphi \wedge \neg\varphi$ hold, against the Law of Non-Contradiction.

Of these options, Priest (2021) suggests that (2) gives a better neutral approach to what an impossible world may be, that is, without considering applications —e.g., to Relevant Logics—, ontology, or the legitimacy of Classical Logic. (3) assumes that Classical Logic is accurate. (4) assumes that the Law of Non-Contradiction necessarily holds, but a dialetheist may disagree. On (1), that something impossible may happen does not imply that it will happen, there may be a world where objects can move faster than the speed of light although they accelerate too slowly ever to reach that speed (Priest 2008, sec. 9.7). In non-normal worlds, logical truths such as $\Box\varphi$, but especially those expressed as $\varphi \dashv \psi$, are not guaranteed to hold. Nonetheless, the possible failure of logical truths is ordinary in Philosophy of Logic, especially when conditionals are at stake. The

Law of Double-Negation Elimination may be valid, $\neg\neg p \multimap p$, but if there is a world where Intuitionistic Logic is right, it may fail, so we cannot hold that $\Box(\neg\neg p \multimap p)$. Thus, non-normal worlds can be considered as worlds where the laws of logic are different from our world, just like impossible physical worlds are states of affairs where the laws of physics are different. However, non-normal worlds are not so logically stable as we might expect. It is easy to see that we cannot hold that $\Box(p \multimap p)$, even when the Principle of Identity, $p \multimap p$, is essential to most logical systems.

Now, if non-normal worlds are worlds where logical truths may fail, in a Multi-Modal Logic based on a Non-Normal Logic, the logical truths of the other logic are also vulnerable to failure. Consider the (K/B) axiom. It is easy to see that, if we accept non-normal worlds, we cannot hold that $\Box(K\varphi \multimap B\psi)$ whatever φ or ψ are. Thus, since (SLO) fails since conditionals fail due to non-normal worlds, the omniscience problem may be a problem of conditionality. That gives an alternative view to the point that omniscience is a problem of the behavior of the operators. There may be circumstances where we ought to know/believe something given a certain knowledge/belief, but we fail to do so. Notice that this strategy can be applied similarly to other combinations, e.g., we might accept a Non-Normal Logic for Alethic Logic meanwhile a normal base is acceptable for Tense Logic, this will affect conditionality but time's structure remains untouched.

Certainly, the problem of adopting only non-normal worlds is that (MLO) is still valid. Adding non-normal epistemic states is a straightforward strategy for the failure of (NK). However, this solution is kind of *ad hoc*, especially if it is not clear what non-normal epistemic states are and how their use may be constrained, a proper interpretation is required.[7] I think that one cause to this effect —although it is not the only one— is the old custom of using possible worlds instead of epistemic states in the semantics. Certainly, the considerations for non-normal worlds can be applied to non-normal epistemic states with a kind of success. Indeed, Hintikka (1979) early tried to use impossible worlds to solve the Logical Omniscience Problem. As Rendsvig *et al.* (2023, sec. 5) put it:

> The basic idea is that an agent may mistakenly count among the worlds consistent with its knowledge, some worlds containing logical contradictions. The mistake is simply a product of the agent's limited resources; the agent may not be in a position to detect the contradiction and may erroneously count them as genuine possibilities.

[7] An anonymous referee has pointed this out to me.

Thus, Hintikka seemed to consider worlds as logical violators. This seems like a forced interpretation. Sometimes we fail to recognize a logical consequence not because we think about contradictions but because we are absent-minded, we do not understand the initial concepts or because our deductive capacity is limited or not trained enough —the deductive capability of an expert differs from that of an amateur. In this regard, adding awareness operators to the language, $A\varphi$, is something promising, Logical Omniscience without them is just about the best an agent can do given her initial information (see Schipper 2015). The framework based on the set E seems more accurate for developing a theory of awareness. Kripke (1965b) originally developed non-normal worlds as a technical resource to get weaker modal logics than $S4$, but the philosophical discussion on their meaning has been a labor of decades; maybe the same history will happen with non-normal epistemic states, being the first step to consider epistemic states as such, not as worlds, after all, analogously, we can think that part of Tense Logic's success is that time structure is at stake, not worlds. Personally, for Epistemic Logic, I think that Plausibility Models are quite good since the models do not have to be based on equivalence relations but on partial orders that allow indicating what an agent considers is most plausible to be the case, and there have been developments on adding awareness operators over this same machinery (see Smets and Velázquez-Quesada 2023, sec. 2.6), for an overview on Plausibility Models, and Velázquez-Quesada 2014, on how Awereness operators may be added). I do think that there are better options to avoid Logical Omniscience. An advantage of fusion semantics is that both logics can have distinct formulations, the most salient case is when one is normal while the other is not. Thus, other techniques to get non-normal Epistemic Logics such as neighborhood semantics or awareness operators are on the table (for neighborhood semantics, see Pacuit 2017).

In the following sections, I will make use of non-normal epistemic states to avoid Logical Omniscience as stated in (MLO) and (SLO). Nevertheless, it is worth saying once more that this is neither the best nor the unique solution, but the easiest from a technical point of view. In the remainder of this paper, our interest will be focused on logical omniscience concerning other logical principles such as the closure for knowledge.

4. *Non-Classical Conditionality for Epistemic Logic*

Granted that Logical Omniscience may be a conditionality issue, perhaps the most promising logical enterprise has to do with the rivals of classical logic. In this section, I would like to explore three logical systems for Epistemic Logic: Conditional Logic, Intuitionistic Logic, and some weak Relevant Logics. For the sake of simplicity, Doxastic Logic will be left aside most of the time. We will work only considering Epistemic Logic.

4.1. Epistemic Logic Based on Conditional Logic

Using Conditional Logic mixed with Epistemic Logic is not so new. I would like to give a historical example. One of the first attempts at developing an Epistemic-Doxastic Logic based on Conditional Logic concerns an argument due to Nozick (1981) in Knowledge Analysis. Gettier (1963) showed that having a true justified belief about p is not sufficient to know p, e.g., a person may truly believe that there is a cop outside her house because she hears the police car sirens, however, the police car may be silent while a kid is playing with a police car toy (Dancy 1985, ch. 3). Thus, many epistemologists have suspected that something is missing to define knowledge. Let $\Box\!\!\rightarrow$ be a counterfactual conditional. Nozick believed that to know something we need a fourth condition:

> *Sensitivity*: If p were false, a would not believe that p, that is, $\neg p \Box\!\!\rightarrow \neg Bp$

Knowledge cannot be accidental, it must prevail and be traceable along different states of affairs. Not many agree that Nozick's idea is correct (see Feldman 2003, pp. 86–90), but it illustrates how Conditional Logic may be applied to Epistemic Logic.[8] Indeed, since contraposition is invalid with $\Box\!\!\rightarrow$, Sosa (1999) has suggested developing the non-equivalent condition of:

> *Safety*: If a were to believe p, p would not be false, that is, $Bp \Box\!\!\rightarrow p$

[8] A reviewer pointed out that the notion of sensitivity is usually rejected from the logical point of view since the distribution of the epistemic operator over conjunction is not valid, something on which Kripke (2011 [1986]) is a mandatory reference. The reviewer also suggested developing this issue given the machinery described below. However, I think that issue, along with its philosophical discussion, deserves another paper. My only intention in mentioning the sensitivity condition is to show how Conditional Logic techniques can be applied to Epistemic Logic.

(For details on Nozick's and Sosa's conditions, see Ichikawa and Steup 2018, sec. 5.1 and 5.2).

Now, in this subsection, I would rather like to discuss an important result in Conditional-Epistemic Logic concerning the failure of the Counterfactual Closure Principle for Knowledge.

$$(\mathrm{K}K \,\Box\!\!\rightarrow)\ K(p \Box\!\!\rightarrow q) \Box\!\!\rightarrow (Kp \Box\!\!\rightarrow Kq)$$

I will leave aside the Logical Omniscience problem since it can be solved by adding non-normal epistemic states.[9] However, the failure of (KK $\Box\!\!\rightarrow$) may have consequences for the Epistemology of Modality.

Although it is too much to ask for an agent, sometimes we can assume that (5K) holds. If an agent is playing poker, she can perfectly know that she does not know which the other player's cards are. Assuming $S5_{\Box}$ and (TK) are accurate, adopting (5K) would also imply accepting (ABF) and (ABCF), and so (KK+). This may be unsatisfactory. After all, what relation would (5K) have with the Closure Principle for Knowledge? The validity of (5K) may not fatally imply the validity of the Closure Principle, at least not one version. It is possible to modify the fusion semantics above described to develop a Conditional-Epistemic Logic, even one based on $S5_K$, where (K$K\Box\!\!\rightarrow$) fails.

As it is standard, we can use the Lewis' (1973) spheres models to deal with counterfactual conditionals (I will make a reconstruction based on Priest 2008, sec. 5.6). A spheres model is a structure $\langle W, \{S_w : w \in W\}, \nu\rangle$. W and ν are as in modal logic (the base logic for alethic operators is $S5_{\Box}$). For each $w \in W$, $w \in S_w \subseteq \ldots \subseteq S_w^n = W$. $[\varphi] = \{w : \nu_w(\varphi) = 1\}$ and $f_\varphi(w) = S_w^i \cap [\varphi]$ such that S_w^i is the smallest sphere of w, i, such that its intersection with $[\varphi]$ is not empty. This last notion gives substance to the idea of closest worlds to w —the most similar worlds to w— where φ holds. The truth conditions for $\Box\!\!\rightarrow$ are:

$$\nu_w(\varphi \Box\!\!\rightarrow \psi) = 1 \text{ iff } f_\varphi(w) \subseteq [\psi]$$

In other words, $\varphi \Box\!\!\rightarrow \psi$ is true in w if all of the closest φ-worlds to w are ψ-worlds.

[9] Nonetheless, as an anonymous referee pointed out, an Epistemic Conditional Logic may also be useful to explore a new kind of Logical Omniscience since $\models$ is monotonic while $\Box\!\!\rightarrow$ is not, the characteristic conditional does not mirror the entailment relation while $\rightarrow$ and $⥽$ do.

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We can use the sphere models as a base for Counterfactual-Alethic Logic while we use $S5_K$ for Epistemic Logic. We only have to do the proper relativization. To keep it simple, we can assume that epistemic states share the same sphere model (whether this constraint gives a stronger logic I do not know, but it will likely do if the logic is weaker than $S5_\Box/S5_K$). Thus, a model would be a structure $\langle W, E, \{\sim_e^\Box : e \in E\}, \{S_w : w \in W\}, \{\sim_w^K : w \in W\}, \nu\rangle$. $[\varphi]$ is now $[\varphi]_e = \{w : \nu_{w/e}(\varphi) = 1\}$ and $f_\varphi(w)$ changes to $f_\varphi(w,e)$, and this may be understood as the class of the closest worlds to w where φ is true in e. In Conditional-Tense Logic, we can evaluate the worlds with the same temporal facts (Sánchez-Hernández 2022c, sec. 4), in Conditional-Epistemic Logic, we check the worlds with the same epistemic facts. The new truth conditions for the counterfactual conditional are:

$$\nu_{w/e}(\varphi \Box\!\!\rightarrow \psi) = 1 \text{ iff } f_\varphi(w,e) \subseteq [\psi]_e$$

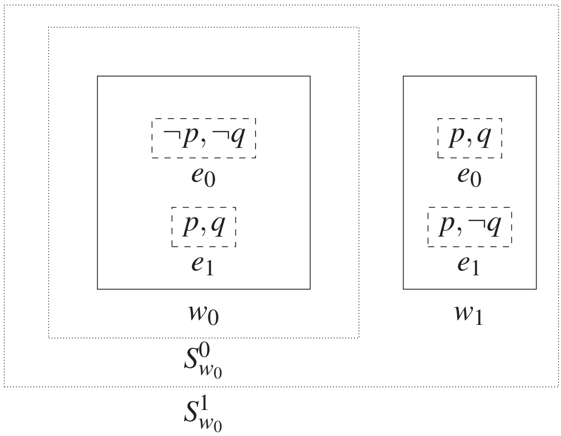
Finally, we can get the Lewis' system $C1$ —VC— and Stalnaker's system $C2$ by adding two more respective constraints, for all $e \in E$:

If $w \in [\varphi]_e$ and $w' \in f_\varphi(w,e)$, then $w = w'$

If $x \in f_\varphi(w,e)$ and $y \in f_\varphi(w,e)$, then $x = y$

The first condition is satisfied if S_{w_0} is a singleton. The second is satisfied if S_{w_0} is a singleton and, for all other S_i, $S_i - S_{i-1}$ is a singleton.

Saying all this, we can proceed to give a countermodel for (K$K\Box\!\!\rightarrow$) in $C2_{\Box\!\rightarrow}/S5_K$ (the spheres are depicted with densely dotted lines):



Both e_0 and e_1 have the same sphere models. However, the point of reference makes the difference, e.g., $f_p(w_0, e_0) = \{w_1\}$ while $f_p(w_0, e_1) = \{w_0\}$. Now, $[p]_{e_0} = [q]_{e_0} = \{w_1\}$, $[p]_{e_1} = \{w_0, w_1\}$, $[q]_{e_1} = \{w_0\}$. $f_p(w_0, e_0) \subseteq [q]_{e_0}$, therefore $p \,\Box\!\!\rightarrow q$ holds in w_0/e_0. $f_p(w_0, e_1) \subseteq [q]_{e_1}$, therefore $p \,\Box\!\!\rightarrow q$ holds in w_0/e_1. Since $p \,\Box\!\!\rightarrow q$ is true in all of the epistemic states of w_0, $K(p \,\Box\!\!\rightarrow q)$ holds in w_0/e_0. Now, in w_1, we have an epistemic model. p holds both in e_0 and e_1 in w_1. Thus, Kp holds in w_1/e_0. However, q fails in w_1/e_1. Hence, Kq fails in w_1/e_0, that is, $\neg Kq$ holds in w_1/e_0. Since $\neg Kp \wedge \neg K\neg p$ holds in w_0, w_1 is the closest world to w_0 where Kp holds, both in e_0 and in e_1. Now, the closest world to w_0 where Kp holds Kq fails, both in e_0 and in e_1. Thus, $Kp \,\Box\!\!\rightarrow Kq$ fails in w_0/e_0, and so $K(p \,\Box\!\!\rightarrow q) \,\Box\!\!\rightarrow (Kp \,\Box\!\!\rightarrow Kq)$. Since S_{w_0} and $S_{w_1} - S_{w_0}$ are singletons, and the relations for K are universal in each world, this is a model for $C_2/S5_K$. Since (K$K\Box\!\!\rightarrow$) fails in $C_2/S5_K$, *a fortiori* it will fail in weaker systems.

Even if $S5_{\Box}/S5_K$ can be acceptable, (KK+) included, the Closure Principle for Knowledge fails when counterfactual conditionals are on stage.

Now, the failure of (K$K\Box\!\!\rightarrow$) is important to the Epistemology of Modality. Over the last two decades, starting with Williamson (2007, ch. 5), the literature has tended to explore the knowledge of alethic necessity as a byproduct of the capacity to state and evaluate counterfactual conditionals (Mallozzi *et al.* 2023, sec. 4.2). It is possible to show that $\Box p$ is equivalent to the following formulas:

(V) $\neg p \Box\!\!\rightarrow p$

(V′) $\neg p \Box\!\!\rightarrow \bot$

(Q) $\forall q(q \Box\!\!\rightarrow p)$

A notable feature of this account is its background evolutionary considerations (Kroedel 2017). Let us crudely define “fitness” as how well-adapted an individual is to his environment, whether we talk about survival or reproduction (for a discussion on this concept, see Rosenberg and Bouchard 2023). Knowing that a proposition is necessary, $K\Box p$, does not seem to improve fitness *prima facie* —it is enough to know p, Kp. However, counterfactual reasoning does seem to improve fitness since evaluating counterfactual conditionals is useful in learning from our mistakes and acting differently in similar situations. An experiment with pilots has shown that those pilots who used counterfactual reasoning after a difficult flight simulation had a better second performance than those who did not (Morris and Moore 2000). Thus, modal reasoning may have come as a byproduct of this more useful ability in survival and planning, e.g., by (V′), $K(\neg p \Box\!\!\rightarrow \bot) \rightarrow K\Box p$. Nevertheless, the failure of closure with $\Box\!\!\rightarrow$ may be important in elucidating whence the fitness of counterfactual reasoning comes from. Suppose our agent is a pilot who, after reflecting a little on her failed simulator test, comes to know that, if there were a problem with the engine, she would need to follow a certain procedure to fix it, $K(p \Box\!\!\rightarrow q)$. Since epistemic closure fails with $\Box\!\!\rightarrow$, knowing the true counterfactual, $K(p \Box\!\!\rightarrow q)$, is not sufficient for her to know that, in a similar situation, she would have to do the procedure, Kq, if she were to know that the engine is failing, Kp, we cannot obtain $Kp \Box\!\!\rightarrow Kq$. Nonetheless, our agent is not hopeless in the use of logical deduction, she could use *modus ponens* with $\Box\!\!\rightarrow$, which requires both Kp and $K(p \Box\!\!\rightarrow q)$ —none of them alone works to know what it is needed to do—:

(KMP$\Box\!\!\rightarrow$) $(K(p \Box\!\!\rightarrow q) \wedge Kp) \Box\!\!\rightarrow Kq$

As we have seen, the equivalence between *modus ponens* and epistemic closure holds when we use $\rightarrow$, it is even valid with $\prec$ under specific circumstances. However, in Conditional-Epistemic Logic, the situation changes. Epistemic closure and *modus ponens* are not equivalent in their counterfactual versions under any circumstance. Thus, given the technical results, if counterfactual reasoning does improve an individual's fitness, we may suggest that fitness improvement may come primarily from using *modus ponens* with the counterfactual conditional rather than from epistemic closure. This may amount to an evolutionary skepticism on the value of epistemic closure.

4.2. Epistemic Logic Based on Intuitionistic Logic

One of the most prolific applications of Multi-Modal Logics in the literature is the analysis of Fitch's Knowability Paradox (Smets and Velázquez-Quesada 2023, sec. 5.2). There is a debate on how to formalize the Paradox, but a standard form is the original one due to Fitch (1963) involving quantification over propositions:

(FKP) $\forall p(p \rightarrow \Diamond Kp) \vdash \forall p(p \rightarrow Kp)$

If it is possible for an agent to know every true proposition, then the agent knows every true proposition. Fitch's Paradox requires a Multi-Modal Logic insofar as alethic and epistemic modalities are involved. Nevertheless, the Paradox has been fruitful for Multi-Modal Logics because some logicians —starting with Williamson (1992)— have tried to analyze it based on Intuitionistic Logic (see Brogaard and Salerno 2019, sec. 3). Nevertheless, Intuitionistic-Epistemic Logic has been a growing field of research in its own right (e.g., see the work of Murai and Sano (2022). In this section, I would like to make a contribution to this literature developing how to modify the fusion semantics to get an Intuitionistic-Epistemic Logic. Again, my concern will be with the Closure Principle for Knowledge. However, I will also see how Intuitionistic Logic avoids Logical Omniscience concerning the definitions of the epistemic operators and a possible critique one can make of (5K). Artemov and Protopopescu (2016) have developed an Intuitionistic Epistemic Logic using the original Brouwer-Heyting-Kolmogorov semantics. However, here we will take for the base the possible worlds semantics developed by Kripke (1965a).

The language for the Intuitionistic-Epistemic Logic is the same as the language for Classical Epistemic Logic. An interpretation for it is a structure $\langle W, E, \{\leq_e : e \in E\}, \{\Psi^K_w : w \in W\}, \nu\rangle$ (notice that $\leq$ substitutes $R^\Box$ since we will not have $\Box$ in the language). The details are the same as before, except that $\{\leq_e : e \in E\}$ is relatively reflexive and transitive —like in $S4_\Box$. The truth conditions for the elements of the language are the following:

$\nu_{w/e}(\varphi \wedge \psi) = 1$	iff	$\nu_{w/e}(\varphi) = \nu_{w/e}(\psi) = 1$
$\nu_{w/e}(\varphi \vee \psi) = 1$	iff	$\nu_{w/e}(\varphi) = 1$ or $\nu_{w/e}(\psi) = 1$
$\nu_{w/e}(\neg\varphi) = 1$	iff	for all $w' \in W$ s. t. $w \leq_e w'$, $\nu_{w'/e}(\varphi) = 0$
$\nu_{w/e}(\varphi \rightarrow \psi) = 1$	iff	for all $w' \in W$ s. t. $w \leq_e w'$, $\nu_{w'/e}(\varphi) = 0$ or $\nu_{w'/e}(\varphi) = 1$
$\nu_{w/e}(K\varphi) = 1$	iff	for all $e \in E$ s. t. $e\Psi^K_w e'$, $\nu_{w/e'}(\varphi) = 1$
$\nu_{w/e}(\widehat{K}\varphi) = 1$	iff	for some $e \in E$ s. t. $e\Psi^K_w e$, $\nu_{w/e'}(\varphi) = 1$

What is left is to characterize the heredity condition. In Kripke's semantics, this is stated as follows:

For all p, if $\nu_w(p) = 1$ and $w \leq w'$, then $\nu_{w'}(p) = 1$

It can be shown that, in propositional logic, the condition extends to any formula of the language (see Priest 2008, footnote to sec. 6.3.5). Hence, for the Multi-Modal case, we will apply the heredity condition to all the language's formulas —including epistemic formulas. Thus

For all φ, if $\nu_{w/e}(\varphi) = 1$ and $w \leq_e w'$, then $\nu_{w'/e}(\varphi) = 1$

Since there are no constraints on Epistemic Logic, the system described can be called I/K_K.

The logic I/K_K has some notable features. To begin with, due to the heredity condition, the Closure Principle for Knowledge holds:

(KK) $K(p \to q) \to (Kp \to Kq)$

This is not surprising. If we had $\Box$ in our language, it is easy to see that the heredity condition validates that $Kp \to \Box Kp$, which is stronger than (ABF), the principle required to get (KK+). Thus, since (NK) is still valid with intuitionistic principles, we have a new version of Logical Omniscience:

(ILO) If $\vdash \varphi \to \psi$ in Intuitionistic-Epistemic Logic, then $\vdash K\varphi \to K\psi$

Again, of course, this may be amended if we add non-normal epistemic states to the semantics. Nontheless, insofar Intuitionistic Logic is a weaker logic than Classical Logic, this kind of Logical Omniscience is weaker.

Nevertheless, Intuitionistic-Epistemic Logic fails to validate the equivalences between Kp and $\widehat{K}p$. Again, this is not surprising considering the behavior of $\forall$ and $\exists$ in Quantified Intuitionistic Logic and that K and $\widehat{K}$ are semantically quantifiers over epistemic states. Certainly, the following holds:

(i) $Kp \to \neg\widehat{K}\neg p$

(ii) $\widehat{K}p \to \neg K\neg p$

However, the following fails:

(iii) $\neg\widehat{K}\neg p \to Kp$

(iv) $\neg K\neg p \to \widehat{K}p$

Consider (iv). It is usually assumed in Epistemic Logic that, if you do not know something, you can consider the epistemic possibility of the negation. This is perfectly reasonable in bounded cases. If you do not know whether a player has the ace of spades, $\neg Kp \wedge \neg K\neg p$, you can conceive both the scenario where he has it and where he does not, $\widehat{K}\neg p \wedge \widehat{K}p$. However, let us consider an open question. Assume an agent, a, does not know that Homer does not talk about the wood horse that Acheans used to raid the city of Troy, $\neg K\neg p$.

Suppose a does know who Homer is although she has not read the *Iliad*. It is epistemically possible for a that Homer does talk about the wood horse that Acheans used to raid the city of Troy, $\widehat{K}p$. On the contrary, suppose a does not know anything concerning Homer and the *Iliad*. It is still true that $\neg K \neg p$, but it is hard to believe that $\widehat{K}p$. Intuitionist Logic makes sense in this kind of case for it was originally developed to deal with situations involving Infinity (Moschovakis 2023). There are indeterminate cases where we do not know certain things, yet the epistemic possibility of the negation is not available.

Consider (iii). The antecedent states that there is a proposition, p, whose negation is epistemically impossible, $\neg\widehat{K}\neg p$. This is not an extraordinary case on a sensitive basis. Although understanding the connection between alethic necessity and knowledge is an elusive issue, alethic impossibility and epistemic impossibility may be closer than expected in some cases:

> That it is *impossible* to divide 23 evenly by 3 explains why no one has ever succeded in figuring out a way to do so, no matter how much mathematics she knows —and why every time someone tries to divide 23 objects evenly into thirds, she fails. None of these efforts could have ever succeded. They all fail because they must: their failure was inevitable. (Lange 2009, p. 6)

Nevertheless, the epistemic impossibility of $\neg p$, $\neg\widehat{K}\neg p$, does not entail that we know its affirmation, Kp (equally, p being epistemically impossible, $\neg\widehat{K}p$, does not entail that we know its negation, $K\neg p$). Suppose it is epistemically impossible for the Goldbach conjecture to be false —on the assumption that it is necessarily true—, that does not entail that we know it is true.

Finally, let us consider the introspection principles. ($4K$) is valid if $\{\Psi_w^K : w \in W\}$ is relatively transitive. On the other hand, ($5K$) fails even if $S5_K$ is at the base. Here is a countermodel for $I/S5_K$ to that effect ($p+$ means that p holds and $p-$ that p fails).

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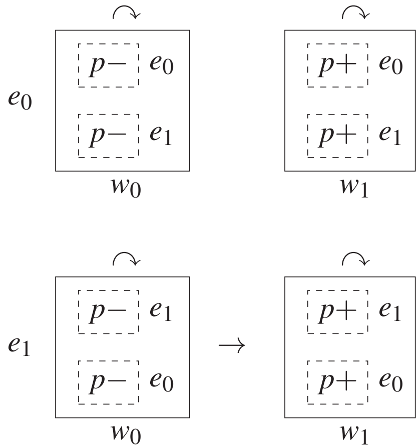
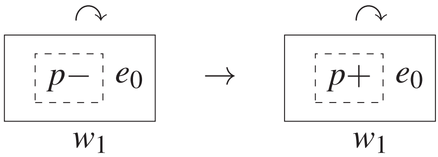


For this case, it is better to work backward. For all e in w_1, p holds, and so Kp both in w_1/e_0 and w_1/e_1. However, $w_0 R_{e_1} w_1$. Thus $\neg Kp$ fails in w_0/e_1, and so $K\neg Kp$ both in w_0/e_1 and w_0/e_0. Now, p fails in w_0/e_1. Thus Kp fails both in e_0 and in e_1 at w_0. However, w_0 is the only accessible world to w_0 in e_0. Thus, $\neg Kp$ holds at w_0/e_0. Therefore, $\neg Kp \rightarrow K\neg Kp$ fails at w_0/e_0.

The failure of (5K) may be natural in the intuitionistic framework. When our discourse is bounded, (5K) is acceptable. In a fair game, say, a friendly poker game, you do not know the other player's cards, $\neg Kp$, and —if you are sensitive— you know about your ignorance, $K\neg Kp$. However, if the discourse is not bounded, you must admit that there is so much you do not know and that you are ignorant about your ignorance. Years before I read the *Divine Comedy*, I did not know who Ugolino della Gherardesca was and I did not know that I ignored it. Thus, this Intuitionistic-Epistemic Logic seems to work well.

Nevertheless, there is an assumption that should be noted. In the countermodel above we assume that $\leq$ is relativized to epistemic states, $\{\leq_e : e \in E\}$. What if that is wrong? If $S5_K$ is at the base and we make $\leq$ absolute, that is, a model is a structure $\langle W, E, \leq, \{\sim^K_w : w \in W\}, \nu\rangle$, then (5$K$) is valid. To see how let us modify the countermodel above with the constraint (in the above model, we needed e_0 and e_1 —although the values were always the same— since $\leq$ changed for one and another; however, since we do not need that difference, we can simply have e_0).



Like before, Kp holds in w_1/e_0, and, since w_0Rw_1, $\neg Kp$ fails in w_0/e_0, and so $K\neg Kp$ in w_0/e_0. However, in w_0/e_0, $\neg Kp$ must hold. Since w_0Rw_0, Kp fails in w_0/e_0. Thus, there must be an epistemic state, e_0, such that p fails in w_0/e_0. So far so good. However, since $\neg Kp$ holds at w_0/e_0 and w_0Rw_1, then Kp must fail in w_1/e_0, but that is a contradiction. Thus, (5K) is valid by *reductio*. When $\leq$ is relative to e, $\neg Kp$ never reaches the world where Kp holds. Nevertheless, the constraint is not enough to deliver Classical Epistemic Logic. It is straightforward to check that this same model is a countermodel where (iv) fails.

4.3. Epistemic Logic Based on Weak Relevant Logics

Although introducing the strict conditional into the game avoids the Material Implication Paradoxes, none of the logics we have reviewed avoids the Paradoxes of Strict Implication:

$$p \wedge \neg p \vDash q$$

$$p \vDash q \vee \neg q$$

In this section, I will develop a minimal Relevant-Epistemic Logic. The logic I will take as a basis is the FDE-based logic N_4 (see Priest 2008, sec. 9.4), which is weaker than the relevant logic B.

A N_4-interpretation has the following structure: $\langle W, W_N, \rho\rangle$. W and W_N are the same in non-normal modal logics. Instead of the usual valuation functions, ν, ρ now indicates that propositional parameters are related to classical truth-values along worlds, that is, where $V = \{0, 1\}$, we have that $\rho \subseteq \mathscr{P} \times W \times V$, so $p\rho_w 1$ intuitively means that p relates to 1 at w. In each world, w, every propositional parameter, p may fall into one of the following valuations: it is true and not false, $p\rho_w 1$ but it is not the case that $p\rho_w 0$; it is false and not true, $p\rho_w 0$ but it is not the case that $p\rho_w 1$; it is both true and false, $p\rho_w 1$ and $p\rho_w 0$ (a truth-value glut); or, finally, it is neither true nor false, it is neither the case that $p\rho_w 1$ nor that $p\rho_w 0$ (a truth-value gap). The truth values for the Boolean connectives are similar to those in classical logic.

$\varphi \wedge \psi\rho_w 1$ iff $\varphi\rho_w 1$ and $\psi\rho_w 1$
$\varphi \wedge \psi\rho_w 0$ iff $\varphi\rho_w 0$ or $\psi\rho_w 0$

$\varphi \vee \psi\rho_w 1$ iff $\varphi\rho_w 1$ or $\psi\rho_w 1$
$\varphi \vee \psi\rho_w 0$ iff $\varphi\rho_w 0$ and $\psi\rho_w 0$

$\neg\varphi\rho 1$ iff $\varphi\rho 0$
$\neg\varphi\rho 0$ iff $\varphi\rho 1$

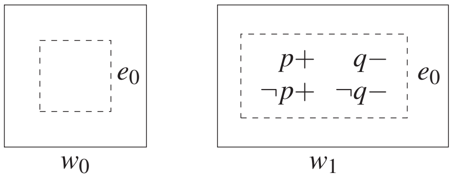
The truth conditions for $\rightarrow$ depend on the nature of w. If $w \in W_N$, the truth conditions for $\rightarrow$ are similar to those for $⥽$ in $S5$ (we still use $\rightarrow$ instead of $⥽$ to keep on with the discussion about conditionality; the material implication can still be defined as $\neg\varphi \vee \psi$, but it does not even satisfy *modus ponens*).

$\varphi \rightarrow \psi\rho_w 1$ iff, for all $w' \in W$, if $\varphi\rho_{w'} 1$, then $\psi\rho_{w'} 1$
$\varphi \rightarrow \psi\rho_w 0$ iff, for some $w' \in W$, $\varphi\rho_{w'} 1$ and $\psi\rho_{w'} 0$

However, if $w \in W - W_N$, the truth value of $\varphi \rightarrow \psi$ is arbitrary, as it were any ordinary propositional parameter (this feature gives as a base a non-normal logic weaker than $S2$ usually known as $S0.5$). Semantic validity is defined by means of truth preservation (whether the premises are only true or they are both true and false) like the non-normal modal logics. N_4 is a relevant logic for every time $\vDash \varphi \rightarrow \psi$ holds, φ and ψ share at least one propositional parameter (Priest 2008, sec. 9.7.9).

We can take N_4 as a base for a Minimal Relevant-Epistemic Logic. The strategy is the same as before. An interpretation for the language is $\langle W, W_N, E, \{\sim_e^{\rightarrow}: e \in E\}, \{\Psi_w^K : w \in W\}, \rho\rangle$. W, W_N, E, and $\{\Psi_w^K : w \in W\}$ are the same as always. $\{\sim_e^{\rightarrow}: e \in E\}$ is the universal accessibility relation for $\rightarrow$ along epistemic states. ρ is now relative to worlds and epistemic states, that is, $\rho \subseteq \mathscr{P} \times W \times E \times V$, so $p\rho_{w/e}1$ intuitively means that p relates to 1 at w/e. All the details are the same as before *mutatis mutandis*. This semantics gets N_4/K_K. To get $N_4/S5_K$ we only need to change $\{\Psi_w^K : w \in W\}$ for $\{\sim_w^K: w \in W\}$. Any system between them is easy to get.

The addition of epistemic operators (even without non-normal epistemic states) does not affect the parameter-sharing condition. This can be shown by considering that $\models K(p \wedge \neg p) \rightarrow K(q \vee \neg q)$ fails even in $N_4/S5_K$. Here is a countermodel to that effect (only w_0 is normal, $\varphi+$ means that φ is related to 1, $\varphi-$ that it is not):



Relevance is accomplished even among epistemic operators.

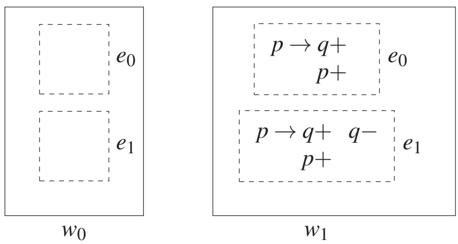
Now, what happens with Logical Omniscience in this framework? (NK) is still valid. This may be unsatisfactory, but adding non-normal epistemic states for it to fail is straightforward. Nonetheless, we still have a weaker Logical Omniscience than Classical Logical Omniscience. The case of the Closure Principle for Knowledge is ambiguous. We have four versions of it:

(i) $\vdash K(p \to q) \to (Kp \to Kq)$
(ii) $\vdash (K(p \to q) \wedge Kp) \to Kq$
(iii) $K(p \to q) \vdash Kp \to Kq$
(iv) $K(p \to q), Kp \vdash Kq$

(i) fails even in $N_4/S5_K$, although (iii) is valid in that system. (iv) is valid even in N_4/K_K. Despite what we could expect, (ii) —*modus ponens* in epistemic context— fails even in $N_4/S5_K$. In a table:

		N_4/K_K	$N_4/S5_K$
(i)	$\vdash K(p \to q) \to (Kp \to Kq)$	×	×
(ii)	$\vdash (K(p \to q) \wedge Kp) \to Kq$	×	×
(iii)	$K(p \to q) \vdash Kp \to Kq$	×	√
(iv)	$K(p \to q), Kp \vdash Kq$	√	√

The $N_4/S5_K$ countermodel for (ii) is the following:[10]



[10] This countermodel has been obtained through a suitable tableau. The full system can be found in Appendix 1.

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It may feel odd to have $p \to q$ and p without q in w_1/e_1. However, the value of $p \to q$ is already arbitrary in non-normal worlds; it does not need to satisfy *modus ponens* to be true. Besides, in the Epistemic Logic literature, it is commonplace to say that some agents may be aware both of $p \to q$ and p, $A(p \to q)$ and Ap, without being aware of q, something achievable with non-normal semantics or Awareness operators. Nonetheless, in the Relevant-Epistemic Logic here developed, the base for Knowledge is normal; it is the relevant base which needs non-normal semantics.

To close this section, it is worth saying that N_4 is a weak logic. The project of developing stronger logics, starting with those with ternary relations, is an open path for anyone willing to do it. The development of proper axiomatic systems is also something I think is worth exploring. Even more, changing the truth conditions for $\to$ properly may even give a first Connexive-Epistemic Logic. There is too much to do with different logics to explore Epistemic Logic.

5. *Conclusions*

Throughout the paper I wanted to show the potential of Multi-Modal Logics in analyzing Logical Omniscience from a conditionality point of view. The basic fusion semantics assumes that there is a difference between possible worlds and epistemic states. Given the results of this paper, I think they are worth considering both philosophically and technically. Multi-Modal Logics give a new framework to consider possible causes of Logical Omniscience, it also opens up the possibility of building up new epistemic systems which may be used in epistemological discussions. We saw that Multi-Modal Logics may vindicate Dretske's notion of penetrating operators and that they allow us to differentiate absolute from relative knowledge. Indeed, the validity of some instances of the Closure Principle for Knowledge requires that some of our knowledge be absolute. Strict Logical Omniscience may fail in two ways. First, if we adopt non-normal worlds, which may fail to validate the logical truths of both logics in consideration. Second, if we adopt non-normal epistemic states, which would invalidate both Material Logical Omniscience and Strict Logic Omniscience. Nevertheless, even in $S2_{\Box}/S2_K$, our agent can use strict *modus ponens*, $(K(p \prec q) \land Kp) \prec Kq$, which is equivalent to the Closure Principle in Classical Logic but it is not in most Multi-Modal Logics. Nonetheless, elucidating the meaning and nature of non-normal epistemic states is left on the agenda. Perhaps the first step is to consider epistemic states as such rather than as worlds, just like in Tense Logic we consider time itself and its structure. Developing Epistemic Logics based on Conditional Logic, Intuitionistic Logic, and Relevant Logic is also possible. In Conditional-Epistemic Logic, the Counterfactual Closure Principle for Knowledge fails even in $C2_{\Box\!\!\rightarrow}/S5_K$. This may be important for the epistemology of modality since the fitness of an individual may depend more on its capacity to apply *modus ponens* rather than the closure of knowledge. Then we saw that Intuitionistic-Epistemic Logic fails to validate some inferences that may carry some omniscience. In particular, we saw that some principles fail when our discourse is not bounded. Finally, we saw how to develop some weak Relevant Epistemic Logics, which avoid Strict Implication Paradoxes in epistemic contexts. Although Multi-Modal Logics are not so new, at least they can be traced back to the early seventies, there is a lot of work to be done with them.

Appendix 1: Tableaux System for N_4/K_K and $N_4/S5_K$

All the Multi-Modal systems in this article have tableaux systems, some of them were developed in Sánchez-Hernández (2022b), and the project was extended in Sánchez-Hernández (2022a). The only exceptions are those for Relevant-Epistemic Logics in sec. 4.3. Thus, in this appendix, I will develop the tableaux systems for N_4/K_K and $N_4/S5_K$ according to their fusion semantics. I will take for granted that the reader is familiar with semantic tableaux.

Let i, j, k, m, n, and o be natural numbers. The nodes of the tableaux for N_4/K_K can have three forms: $\varphi, +i/m$ means that φ relates to 1 in the world i of the epistemic state m, that is, $\varphi\rho_{w_i/e_m}1$; $\varphi, -i/m$, that it is not the case that $\varphi\rho_{w_i/e_m}1$; and $m\Psi^K_i n$, that the epistemic state m relates epistemically with the state n under the world i. To differentiate whether a number is for a world or an epistemic state, we can underline the numbers for epistemic states, e.g., $\underline{3}\Psi^K_2\underline{4}$. An initial list is made up by a node of the form $\varphi, +i/m$ for every premise (if there is any) and a node of the form $\psi, -i/m$ for the conclusion. A branch is *closed*, $\otimes$, if both $\varphi, +i/m$ and $\varphi, -i/m$ appear on it; otherwise, it is *open*. A tableau is *closed* if all of its branches are closed; otherwise, it is open.

The tableaux rules for $\wedge$, $\vee$, and $\neg$ are the following

$$\begin{array}{c} \varphi \wedge \psi, +i/m \\ \downarrow \\ \varphi, +i/m \\ \psi, +i/m \end{array} \qquad \begin{array}{ccc} & \varphi \wedge \psi, -i/m & \\ & \swarrow \quad \searrow & \\ \varphi, -i/m & & \psi, -i/m \end{array}$$

$$\begin{array}{ccc} & \varphi \vee \psi, +i/m & \\ & \swarrow \quad \searrow & \\ \varphi, +i/m & & \psi, +i/m \end{array} \qquad \begin{array}{c} \varphi \vee \psi, -i/m \\ \downarrow \\ \varphi, -i/m \\ \psi, -i/m \end{array}$$

$$\begin{array}{c} \neg(\varphi \wedge \psi), \pm i/m \\ \downarrow \\ \neg\varphi \vee \neg\psi, \pm i/m \end{array} \qquad \begin{array}{c} \neg(\varphi \vee \psi), \pm i/m \\ \downarrow \\ \neg\varphi \wedge \neg\psi, \pm i/m \end{array}$$

$$\begin{array}{c} \neg\neg\varphi, \pm i/m \\ \downarrow \\ \varphi, \pm i/m \end{array}$$

The $\pm$ indicates that the extension does not change the sign.

The rules for the epistemic operators come in groups. First, we have the equivalence rules:

$$\begin{array}{c} \neg K\varphi, \pm i/m \\ \downarrow \\ \widehat{K}\neg\varphi, \pm i/m \end{array} \qquad \begin{array}{c} \neg \widehat{K}\varphi, \pm i/m \\ \downarrow \\ K\neg\varphi, \pm i/m \end{array}$$

Then we have the rules for true operators:

$$\begin{array}{c} K\varphi, +i/m \\ m\Psi_i^K n \\ \downarrow \\ \varphi, +i/n \end{array} \qquad \begin{array}{c} \widehat{K}\varphi, +i/m \\ \downarrow \\ m\Psi_i^K o \\ \varphi, +i/o \end{array}$$

The rules are similar to those for $\Box$ and $\Diamond$ in modal tableaux. For $K\varphi$, we need the nodes above the line, where n is any number such that $m\Psi_i^K n$ is on the branch; for $\widehat{K}\varphi$, we extend the branch with two nodes, where o is a new number on the branch. Finally we have the rules for untrue operators:

$$\begin{array}{c} K\varphi, -i/m \\ \downarrow \\ m\Psi_i^K o \\ \varphi, -i/o \end{array} \qquad \begin{array}{c} \widehat{K}\varphi, -i/m \\ m\Psi_i^K n \\ \downarrow \\ \varphi, -i/n \end{array}$$

The rules are similar to those for true operators, but the $-$ signs in the extensions are important.

If we use $S5_K$, the rules for epistemic operators are the same except that we suppress the line for the accessibility relation.

Finally, we have the rules for $\to$:

$$\begin{array}{ccc} & \varphi \to \psi, +0/m & \\ & \swarrow \quad \searrow & \\ \varphi, -j/m & & \psi, +j/m \end{array} \qquad \begin{array}{c} \varphi \to \psi, -0/m \\ \downarrow \\ \varphi, +k/m \\ \psi, -k/m \end{array}$$

The rules are applied only when $i = 0$. For true $\to$, we apply the rule for every j on the branch; for untrue $\to$, k is a new world number on the branch.

Notice that, in every rule, at least one of the numbers is still the same. The rules for epistemic operators do not change the world index, nor do the rules for $\to$ change the epistemic state index.

To illustrate the method, here is the N_4/K_K-tableau for $K(p \to q), Kp \vdash Kq$:

$$\begin{array}{ccc} & K(p \to q), +0/\underline{0} & \\ & Kp, +0/\underline{0} & \\ & Kq, -0/\underline{0} & \\ & \underline{0}\Psi_0^K\underline{1} & \\ & q, -0/\underline{1} & \\ & p \to q, +0/\underline{1} & \\ & p, +0/\underline{1} & \\ & \swarrow \quad \searrow & \\ p, -0/\underline{1} & & q, +0/\underline{1} \\ \otimes & & \otimes \end{array}$$

On the other hand, here is the tableau for $\vdash (K(p \to q) \wedge Kp) \to Kq$ in $N_4/S5_K$:

$$\begin{array}{cc} (K(p \to q) \wedge Kp) \to Kq, - + 0/\underline{0} & \\ K(p \to q) \wedge Kp, +1/\underline{0} & \\ Kq, -1/\underline{0} & \\ K(p \to q), +1/\underline{0} & \\ Kp, +1/\underline{0} & \\ q, -1/\underline{1} & \\ p \to q, +1/\underline{0} & \\ p \to q, +1/\underline{1} & (*) \\ p, +1/\underline{0} & \\ p, +1/\underline{1} & \end{array}$$

This tableau is complete, we cannot apply the rule for true $\to$ to the line with (*) to close the branch (and so the tableau) with the nodes for $p, -1/\underline{1}$ and $q, +1/\underline{1}$. The rules for $\rightarrow$ are only for the world 0, the only normal world.

A countermodel can be read off through an open branch. For every i and m on the branch, $w_i \in W$ and $e_m \in E$. W_N is always $\{w_0\}$, as so is $\{\sim_e^{\rightarrow}: e \in E\}$. If $m\Psi_i^K n$ is on the branch, $e_m \Psi_{w_i}^K e_n \in \{\Psi_w^K : w \in W\}$ (if we are working on $S5_K$, $\{\sim_w^K: w \in W\}$). Finally, for all propositional parameters, if $p, +i/m$ is on the branch, $p\rho_{w/e}1$; and, if $p, -i/m$ is on the branch, $p\rho_{w/e}1 \notin \rho$. The countermodel for $\vdash (K(p \rightarrow q) \wedge Kp) \rightarrow Kq$ is: $W = \{w_0, w_1\}$; $E = \{e_0, e_1\}$; $W_N = \{w_0\}$; $\{\sim_e^{\rightarrow}: e \in E\}$; $\{\sim_w^K: w \in W\}$; $p \rightarrow q\rho_{w_1/e_0}1$, $p \rightarrow q\rho_{w_1/e_1}1$, $p\rho_{w_1/e_0}1$, $p\rho_{w_1/e_1}1$, but $q\rho_{w_1/e_1}1 \notin \rho$. This model has been depicted in sec. 4.3.

The tableaux systems are sound and complete according to their semantics. The proof is left to the reader.

Appendix 2: Non-Classical Conditionality for Deontic Logic

The techniques reviewed in this paper can also be applied to Deontic Logic. As a first approach, if we were to base Deontic Logic on Intuitionistic Logic (like in section 5.2), a legal loophole would not imply permission, for $\neg O \neg q$ does not entail Pq. Nevertheless, in my opinion, the most interesting application concerns the discussion of Hume's Law, which states that *ought*-sentences cannot be derived from *is*-sentences. David Hume questioned the plausibility of the validity of is-ought inferences in his *Treatise on Human Nature* (2000 [1739], III, 1, i):

> In every system of morality, which I have hitherto met with, I have always remarked that the author proceeds for some time in the ordinary way of reasoning, and establishes the being of God, or makes observations concerning human affairs; when of a sudden I am surprised to find, that instead of the usual copulations, *is*, and *is not*, I meet with no proposition that is not connected with an *ought*, or an *ought not*. This change is imperceptible; but is, however, of the last consequence. For as this *ought*, or *ought not*, expresses some new relation or affirmation, 'tis necessary that it should be observed and explained; and at the same time that a reason should be given, for what it seems altogether inconceivable, how this new relation can be a deduction from others, which are entirely different from it.

This passage has been interpreted in many and inconsistent ways, some of them without considering that Hume was not denying the plausibility of an ethical theory, being more fair to say that he was critical of ethical theories based on rationalism instead of emotivism (Thompson 2022, sec. 5.1). There is so much to say here. Indeed, some philosophers have done a great job analyzing the is-ought inferences from the logical point of view. In this appendix, I want to show some considerations we can formulate from the multi-modal point of view.

Arthur Prior (1960) gave the first logical analysis of is-ought inferences. He believed that we can distinguish between *normative* sentences, namely, those in the scope of O, and *descriptive* sentences, those free of O and P. The distinction is clear in simple cases. $q \wedge r$ is descriptive; $O(q \wedge r)$, normative. However, the problem arises when mixed sentences occur, say, $q \wedge Or$. Given this ambiguity, Prior came to a paradox. Consider the following inferences:

(i) $q \vdash q \vee Or$

(ii) $\neg q, q \vee Or \vdash Or$

Due to the Law of Excluded Middle, $q \vee Or$ is either normative or non-normative, that is, descriptive. If it is normative, (i) is an is-ought valid inference. If it is descriptive, (ii) is an is-ought valid inference. Thus, in either case, it is legitimate to derive ought-sentences from is-sentences. Notice that Prior's Paradox can also be derived from the following:

(iii) $\neg q \vdash q \rightarrow Or$

(iv) $q, q \rightarrow Or \vdash Or$

There have been many responses to Prior's Paradox. Kutschera (1977) has suggested that Hume's Law is valid without mixed sentences. However, logicians like Åqvist (2002) emphasize the importance of mixed sentences to avoid Material Implication Paradox. Notice that (iii) does not hold with strict implication, $\neg q \nvdash q \mathrel{-\!\!3} Or$, although Prior Paradox stands still due to the validity of (i) and (ii).

Perhaps the most salient logical analysis proposal is due to Schurz (1994). Schurz calls Kutschera's point of view:

> *The Special Hume Thesis*: No purely normative sentence that is not logically true is deducible from any consistent set of purely descriptive sentences.

The constraints immediately avoid inferences such as $q \wedge \neg q \vdash Or$ and $q \vdash O(r \rightarrow s) \rightarrow (Or \rightarrow Os)$. A stronger statement is:

> *The General Hume Thesis* No purely normative or mixed sentence that is not logically true is deducible from any consistent set of purely descriptive sentences and possibly mixed sentences.

To achieve this thesis, Schurz develops the notion of O-relevance. Given a deduction, $\Sigma \vdash \varphi$ such that φ is normative or a mixed sentence, the conclusion is O-irrelevant iff every parameter in the scope of φ can be uniformly substituted *salva veritate* in the deduction. On these bases, Schurz states the General Hume Thesis as follows:

> For every deduction, $\Sigma \vdash \varphi$, with purely descriptive premises and a possibly mixed conclusion, φ, φ is an O-irrelevant conclusion from Σ.

None from (i) to (iv) is O-relevant. Consider (iv), it can become $q, q \to O\psi \vdash O\psi$ for any arbitrary ψ without problems. However, while the inference using "if global warming continues, we ought to reduce our emissions of greenhouse effect gases" is plausible, the inference using "if global warming continues, we ought to go to a Taylor Swift concert" is not acceptable. Nevertheless, both inferences are valid. The notion of O-relevance shows how is-ought inferences can be irrelevant, and, therefore, unacceptable. Is-sentences are about our world, ought-sentences are about our moral duties; the intuition is that they must remain separated insofar as they talk about different things.

Now, a virtue of Schurz's development is that it is an instance of a broader investigation on relevance in logic and philosophy of language (Schurz 2013), so the response to the problem raised by Hume is not *ad hoc*. He is not the only one to see Hume's Law as an instance of a broader issue. Russell (2022) has also suggested that Hume's Law is an instance of something she calls the Limited General Barrier Theorem, which states that there are some barriers to inferences that we cannot pass due to the nature of the premises and the conclusion, like the inference from past cases to future cases or from non-indexical sentences to indexical sentences (it is worth saying that Russell also explores serious multi-modal systems and their philosophical issues). Nevertheless, Schurz's project differs from those of Relevant Logics developed by American and Australian logicians as he mentions in a note. Nonetheless, these mainstream relevant systems also give a different response to Prior's Paradox, which strikes me insofar as is obvious yet no one —I am aware of— has pointed it out: in Relevant Logics, say, like a modification of those in sec. 5.3 (or a stronger one, say R), neither (ii) —Disjunctive Syllogism— nor (iii) —Material Implication Paradox— are valid; thus, in a Deontic Logic based on a Relevant Logic, Prior's Paradox is not derivable.

Now, there is an easy way to violate Hume's Law if we accept some bridge principles (for discussion, see Schurz 2010). Let us consider:

(BP1) $\Box q \to Oq$

(BP2) $Oq \to \Diamond q$

Although there is no official name for (BP1), (BP2) —"ought imply may"— is commonly known as Kant's Law, for he suggests it in his *Critique of Pure Reason* (B576). Neither is O-irrelevant, e.g., we cannot have $\Box q \to Os$ from (BP1). Hence, the General Hume Thesis fails. Even more, the counterpositive of (BP2), $\neg\Diamond q \to \neg Oq$, is an is-ought inference which may count as valid (nevertheless, some philosophers think Hume's Law and Kant's Law are not opposites, see Spielthenner 2017). Another notable feature of these principles is that, along with (4$\Box$) —$\Box q \to \Box\Box q$—, (BP1) makes O a fully penetrating operator according to Dretske's terminology. Here is an axiomatic proof:

(I)	$\Box q \to Oq$	(BP1)
(II)	$\Box(q \to s) \to O(q \to s)$	(US), $[q/q \to s]$ to (I)
(III)	$O(q \to s) \to (Oq \to Os)$	(KO)
(IV)	$\Box(q \to s) \to (Oq \to Os)$	Hip. Syl. by (II) and (III)
(V)	$\Box(\Box(q \to s) \to (Oq \to Os))$	(N$\Box$) to (IV)
(VI)	$\Box\Box(q \to s) \to \Box(Oq \to Os)$	(KO) to (V)
(VII)	$\Box q \to \Box\Box q$	(4$\Box$)
(VIII)	$\Box(q \to s) \to \Box\Box(q \to s)$	(US), $[q/q \to s]$, to (VII)
(IX)	$\Box(q \to s) \to \Box(Oq \to Os)$	Hip. Syl. by (VIII) and (VI)
(X)	$\Box(\Box(q \to s) \to \Box(Oq \to Os))$	(NO) to (IX)
(XI)	$(q \strictif s) \strictif (Oq \strictif Os)$	Definition of $\strictif$ to (X)

Now, let $q \strictiff s =_{def} (q \strictif s) \wedge (s \strictif q)$. Then we can show that $(q \strictiff s) \strictif (Oq \strictiff Os)$. If we were able to define what "Good" *must* mean, $g \strictiff p$, then we would also be able to know what we ought to do logically if we ought to be good, $Og \strictiff Op$. Nevertheless, G.E. Moore (1903) believed that in ethics it is not valid to assert the Naturalistic Fallacy, that is, given any natural property, P, we cannot state that it is Good, G, on the bases that it is natural. Some philosophers also call Naturalistic Fallacy the Definist Fallacy insofar that, given any purported definition of "Good", we can always ask whether it *is really* good (1999, p. 583, see also Rachels 1990, p. 69). Thus, there is no natural property, P, such that it must imply that its carrier is Good, G, that is, for any natural property we cannot state that $\forall x(Px \dashv\!\! 3\, Gx)$; even more, although $\forall x((Px \dashv\!\! 3\, Gx) \dashv\!\! 3 (OPx \dashv\!\! 3\, OGx))$ is a valid principle, we will never have what it is needed to state, for any P, $\forall x(OPx \dashv\!\! 3\, OGx)$. This is worth considering, but I will leave it here in this paper. Nevertheless, I think the logical analysis here suggests where Hume's Law ends and the issue of Moore's Naturalistic Fallacy begins. This may be helpful to those who have had problems —me included— to sharply understand where the distinction lies between Hume's ideas and Moore's ideas.

To finish this appendix, I would like to say that semantics that validate (BP1) and (BP2) are available. Indeed, there are two options, depending on whether alethic and deontic modalities share the same realm. If they share the same realm, an interpretation can be the structure $\langle W, R^{\Box}, R^{O}, \nu\rangle$. $R^{\Box}$ and R^{O} are both subsets from $W \times W$. ν is the same as usual. To get (BP1), we need that $R^{O} \subseteq R^{\Box}$. To get (BP2), we need the same constraint besides requiring that R^{O} to be serial —for every $w \in W$, there is a w' such that $wR^{O}w'$. If the operator does not share the same realm, then an option would be the following structure: $\langle W, E, \{R^{\Box}_{e} : e \in E\}, \{R^{O}_{w} : w \in W\}, \nu\rangle$. Notice these semantics are essentially the same as those for $T_{\Box}/T_{K}/D^{*}_{B}$ in section 1. The constraints needed are the following. To get (BP1), we need that, for all $e, e' \in E$ and for all $w \in W$, if $eR^{O}_{w}e'$, then $wR^{\Box}_{e}e'$. To get (BP2), we need the same constraint besides requiring for $\{R^{O}_{w} : w \in W\}$ to be relatively serial, that is, for all $w \in W$ and $e \in E$, there is an $e' \in E$ such that $eR^{O}_{w}e'$. The acceptance or rejection of the validity of is-ought inferences can lie in reflecting on these semantics. It may only guide us. I think is-ought inferences are one of the most complex and interesting issues in metaethics since we are discussing how our knowledge of the world may guide us ethically speaking. I hope the logical point of view I have developed here sheds some light on the matter.

Appendix 3: Logical principles abbreviations in the paper

1. Some Multi-Modal Logics

(PC)	All the tautologies, $\vdash \varphi$, of classical propositional calculus
(US)	If $\vdash \varphi$, p is part of φ, and φ' is the same as φ except that ψ uniformly substitutes p, $[p/\psi]$, then $\vdash \varphi'$

(MP)	If $\vdash \varphi \to \psi$ and $\vdash \varphi$, then $\vdash \psi$
(K$\Box$)	$\Box(p \to q) \to (\Box p \to \Box q)$
(KK)	$K(p \to q) \to (Kp \to Kq)$
(KB)	$B(p \to q) \to (Bp \to Bq)$
(N$\Box$)	If $\vdash \varphi$, then $\vdash \Box\varphi$
(NK)	If $\vdash \varphi$, then $\vdash K\varphi$
(NB)	If $\vdash \varphi$, then $\vdash B\varphi$
(T$\Box$)	$\Box p \to p$
(TK)	$Kp \to p$
(DB)	$Bp \to \neg B \neg p$
(K/B)	$Kp \to Bp$
(4$\Box$)	$\Box p \to \Box\Box p$
(5$\Box$)	$\Diamond p \to \Box\Diamond p$
(4K)	$Kp \to KKp$
(5K)	$\neg Kp \to K\neg Kp$
(4B)	$Bp \to BBp$
(5B)	$\neg Bp \to B\neg Bp$
(B/BK)	$Bp \to BKp$
(B/KB)	$Bp \to KBp$
(ABF)	$K\Box p \to \Box Kp$
(ABCF)	$\Box Kp \to K\Box p$

2. On the Strict Closure Principle

(K$\Box$+)	$\Box(p \multimap q) \multimap (\Box p \multimap \Box q)$
(K$\Box\pm$)	$(p \multimap q) \multimap (\Box p \multimap \Box q)$
(KK+)	$K(p \multimap q) \multimap (Kp \multimap Kq)$
(KB+)	$B(p \multimap q) \multimap (Bp \multimap Bq)$

3. Logical Omniscience and Non-Normal Multi-Modal Logics

(MLO)	If $\vdash \varphi \to \psi$, then $\vdash K\varphi \to K\psi$
(SLO)	If $\vdash \varphi \multimap \psi$, then $\vdash K\varphi \multimap K\psi$
(SMP)	If $\vdash \varphi \multimap \psi$ and $\vdash \varphi$, then $\vdash \psi$
(QKK+)	$\Box K(p \to q) \multimap (Kp \multimap Kq)$
(CN$\Box$)	If $\vdash \Box\varphi$, then $\vdash \varphi$
(KSMP)	$(K(p \multimap q) \land Kp) \multimap Kq$

4. Non-Classical Conditionality for Epistemic Logic

Subsection 4.1. Epistemic Logic Based on Conditional Logic

Sensitivity: If p were false, a would not believe that p, that is, $\neg p \,\Box\!\!\rightarrow \neg Bp$

Safety: If a were to believe p, p would not be false, that is, $Bp \,\Box\!\!\rightarrow p$

(K$K\,\Box\!\!\rightarrow$) $K(p \,\Box\!\!\rightarrow q) \,\Box\!\!\rightarrow (Kp \,\Box\!\!\rightarrow Kq)$

(V) $\neg p \,\Box\!\!\rightarrow p$

(V’) $\neg p \,\Box\!\!\rightarrow \bot$

(Q) $\forall q(q \,\Box\!\!\rightarrow p)$

(KMP$\,\Box\!\!\rightarrow$) $(K(p \,\Box\!\!\rightarrow q) \wedge Kp) \,\Box\!\!\rightarrow Kq$

4.2. Epistemic Logic Based on Intuitionistic Logic

(FKP) $\forall p(p \rightarrow \Diamond Kp) \vdash \forall p(p \rightarrow Kp)$

(ILO) If $\vdash \varphi \rightarrow \psi$ in Intuitionistic-Epistemic Logic, then $\vdash K\varphi \rightarrow K\psi$

(i) $Kp \rightarrow \neg\widehat{K}\neg p$

(ii) $\widehat{K}p \rightarrow \neg K\neg p$

(iii) $\neg\widehat{K}\neg p \rightarrow Kp$

(iv) $\neg K\neg p \rightarrow \widehat{K}p$

4.3. Epistemic Logic Based on Weak Relevant Logics

(i) $\vdash K(p \rightarrow q) \rightarrow (Kp \rightarrow Kq)$

(ii) $\vdash (K(p \rightarrow q) \wedge Kp) \rightarrow Kq$

(iii) $K(p \rightarrow q) \vdash Kp \rightarrow Kq$

(iv) $K(p \rightarrow q), Kp \vdash Kq$

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