FIRST-ORDER SEMANTICS FOR HIGHER-ORDER LANGUAGES*

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In this article, expansions of Post-systems ([1]-[3]) are defined and used to construct semantic systems in which quantification over individuals, but not over classes, is allowed. The metalanguage used in defining the concept of logical truth for object languages is a first-order language, whereas the object languages are languages of an arbitrary higher order. The construction of such semantic systems shows that even platonistic languages can have a nonplatonistic foundation. The semantic systems are not defined for specific languages; instead, we define a general concept of language (ν-language) relative to a class Z of constructive ordinals ($\nu \varepsilon Z$). The usual languages of classical first-order logic, simple theory of types and ramified theory of types are examples of such languages.

1. *Formal definition systems*. a_1, a_2, etc. are *atomic names*, x, y, z, x_i, y_i, z_i ($i \geq 1$) are *variables*. *Complex names* (or *words*) are objects $\beta_1 \beta_2 \ldots \beta_n$ ($n \geq 1$), where the β_i are atomic names. *Terms* are objects $\beta_1 \beta_2 \ldots \beta_n$ ($n \geq 1$), each β_i being an atomic name or a variable (e.g. $a_1a_2a_1$ is a complex name and a term, while a_1xya_2 is a term but not a complex name.) p_i^n ($n \geq 1$, $i \geq 1$) is an n-ary *predicate letter*, and for terms α_i, $p_i^n(\alpha_1, \ldots, \alpha_n)$ ($n \geq 1$, $i \geq 1$) is an *atomic formula*. Atomic formulas and objects of the form $\Psi_1, \ldots, \Psi_r \Longrightarrow \Psi$, in which $\Psi_1, \ldots, \Psi_r$, Ψ are atomic formulas, are called *formulas* ($r \geq 1$). An object $[a_1, \ldots, a_k; \gamma_1, \ldots, \gamma_p]$ $\Sigma_1, \ldots, \Sigma_h$

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($k \geq 1$, $p \geq 1$, $h \geq 1$) in which $\Sigma_1, \ldots, \Sigma_h$ are formulas, $\gamma_1, \ldots, \gamma_p$ are all the predicate letters contained in $\Sigma_1, \ldots, \Sigma_h$ and $a_1, \ldots, a_k$ are at least all the atomic names appearing in $\Sigma_1, \ldots, \Sigma_h$ is called a *formal definition system*. We call $a_1, \ldots, a_k$ the *alphabet* and $\gamma_1, \ldots, \gamma_p$ the *predicate letter stock* of the formal definition system.

2. *Semi-formal definition systems.* Atomic names, variables, complex names, terms, n-ary predicate letters and atomic formulas are defined as for the formal definition systems of §1. The concept of formula is then extended as follows:

(a) If $p_i^n(\alpha_1, \ldots, \alpha_n)$ is an atomic formula containing at least m ($m \geq 1$) different variables $\xi_1, \ldots, \xi_m$, then $\Lambda \xi_1, \ldots, \xi_m \, p_i^n(\alpha_1, \ldots, \alpha_n)$ is a *universal formula* (e.g. $\Lambda x, y p_1^2(xa_1, y)$; in this formula each ξ_i is *bound*.)

(b) Atomic formulas, universal formulas and objects of the form $\Psi_1, \ldots, \Psi_r \Longrightarrow \Psi$ ($r \geq 1$) with atomic *or* universal Ψ_i ($1 \geq i \geq r$) and atomic Ψ are called *formulas* subject to the following condition: if Ψ_i is a universal formula $\Lambda \xi_1, \ldots, \xi_m \, p_j^n(\alpha_1, \ldots, \alpha_n)$, then any occurence of the bound variables $\xi_1, \ldots, \xi_m$ in the remaining formulas $\Psi_1, \ldots, \Psi_r$, Ψ is also bound (e.g. $\Lambda x p_1^1(x) \Longrightarrow p_1^1(y)$ is a formula but $\Lambda x p_1^1(x) \Longrightarrow p_2^1(x)$ is not a formula).

An object $[a_1, \ldots, a_k; \gamma_1, \ldots, \gamma_p] \; \Sigma_1, \ldots, \Sigma_h$ ($k \geq 1$, $p \geq 1$, $h \geq 1$) in which $\Sigma_1, \ldots, \Sigma_h$ are formulas, but not universal formulas, $\gamma_1, \ldots, \gamma_p$ are all predicate letters appearing in $\Sigma_1, \ldots, \Sigma_h$ and $a_1, \ldots, a_k$ are at least all atomic names occurring in $\Sigma_1, \ldots, \Sigma_h$ is called a *semi-formal definition system*. $\Sigma_1, \ldots, \Sigma_h$ are called *axioms* of the system. Hence all formal definition systems are also semi-formal definition systems, but not conversely. Semi-formal definition systems differ from formal definition systems in that the former may contain universal quantifiers. Universal formulas do not appear in semi-formal definition systems as axioms, but they may occur within an axiom to the left of the arrow $\Longrightarrow$ (e.g. $p_1^n(\alpha_1, \ldots, \alpha_n) \Longrightarrow \Lambda x p_1^m (\beta_1, \ldots, \beta_m)$ is not a formula, and, thus, not an axiom of a semi-formal definition system). Formal and semi-formal definition systems are called *definition systems.* Let D be a definition system with the alphabet $a_1, \ldots, a_k$, we then call all complex names containing only atomic names of the list $a_1, \ldots, a_k$ *complex names of D* and we call all terms containing only atomic names of D *terms of D.* Finally we call those formulas containing only atomic names and predicate letters of D *formulas of D.*

3. *Proof concepts for formal and semi-formal definition systems.* Let Z be a class of recursive ordinals with $<$ as order relation, and assume that transfinite induction is valid for Z, i.e. for any property H and α, β, γ varying through Z: any element α of Z has the property H, in symbols $H(\alpha)$, if (for any γ), $H(\gamma)$ if $H(\beta)$ for all $\beta < \gamma$. The class Z must be suitably chosen according to the intended application of the definition system. We do not consider here the question of convincing oneself of the validity of transfinite induction for Z. The applications discussed below require only relatively small classes of ordinals. In fact, the class of all ordinals $< \epsilon_0$, for which transfinite induction can be proved in elementary arithmetic, would suffice for our purposes.

Let D be a *definition system*, and let $\Psi_1, \ldots, \Psi_n, \Psi$ ($n \geq 1$) be formulas of D. We then say that Ψ is the result of *substitution* in $\Psi_1, \ldots, \Psi_n$ D ($n \geq 1$) iff Ψ is obtained from a Ψ_i ($1 \leq i \leq n$) by replacing all free occurrences of at least one variable of Ψ_i by a complex name of D (different free variables may be replaced by different complex names).

We say that Ψ results from $\Psi_1, \ldots, \Psi_n$ ($n \geq 2$) in D by *modus ponens* iff at least one Ψ_i ($1 \leq i \leq n$) has the form $\Phi_1, \ldots, \Phi_p \Longrightarrow \Psi$ ($p \geq 1$) and each $\Phi_j (1 \leq j \leq p)$ is a Ψ_i for some i ($1 \leq i \leq n, p \leq n\text{-}1$).

$\Psi_1, \ldots, \Psi_n$ ($n \geq 1$) is called a *formal proof of* Ψ_n *in D* (or *formal proof in D*) iff each Ψ_i ($1 \leq i \leq n$) is an axiom of D or results from $\Psi_1, \ldots, \Psi_{i-1}$ by substitution or modus ponens in D.

Formal proofs in D are also called O-proofs (or *O-inductive proofs*) in D.

$\Psi_1, \ldots, \Psi_n$ is called a *ν-proof of Ψ_n in D* (or *ν-inductive proof of Ψ_n in D*) iff each Ψ_i $(1 \leq i \leq n)$ is either an axiom of D, or the result of a substitution in or modus ponens from $\Psi_1, \ldots, \Psi_{i-1}$ in D, or a universal formula $\Lambda \xi_1 \ldots \Lambda \xi_m \Phi$ with m variables $\xi_1, \ldots, \xi_m$ $(m \geq 1)$ such that each substitution formula of Φ (i.e. each formula obtained by substitution in Φ in D) is μ-provable for an ordinal $\mu < \nu$ in D $(\mu \varepsilon Z, \nu \varepsilon Z)$. From the definition, it follows that for $\mu < \nu$ each μ-proof in D is also a ν-proof in D. The formal (O-inductive) proof concept is mechanically decidable but not the more general concepts of 1-inductive proof, 2-inductive proof, etc.

Let $\sigma(\alpha_1, \ldots, \alpha_n)$ be a formula of D. By $D \overset{\nu}{\vdash} \sigma(\alpha_1, \ldots, \alpha_n)$ or simply $\sigma(\alpha_1, \ldots, \alpha_n)$ in ${}^{\nu}D$ we mean that $\sigma(\alpha_1, \ldots, \alpha_n)$ is *ν-provable* in D, i.e., there is a ν-proof of $\sigma(\alpha_1, \ldots, \alpha_n)$ in D $(\nu \varepsilon Z)$.

4. *Decidable, complementary, disjunct.* Call an n-ary predicate letter σ *decidable in ${}^{\nu}D$* iff there are formal definition systems D_1, D_2 such that for each n-tuple $\alpha_1, \ldots, \alpha_n$ of complex names of D: $\sigma(\alpha_1, \ldots, \alpha_n)$ in ${}^{\nu}D$ if and only if $\sigma(\alpha_1, \ldots, \alpha_n)$ in ${}^{0}D_1$, and not $\sigma(\alpha_1, \ldots, \alpha_n)$ in ${}^{\nu}D$ if and only if $\tau(\alpha_1, \ldots, \alpha_n)$ in ${}^{0}D_2$ for some n-ary predicate letter τ of D_2.

Let σ and $\overline{\sigma}$ be binary predicate letters of D. We call $\overline{\sigma}$ *the complement of σ in ${}^{\nu}D$* iff for each pair α, β of complex names of D: $\overline{\sigma}(\alpha, \beta)$ in ${}^{\nu}D$ if and only if not $\sigma(\alpha, \beta)$ in ${}^{\nu}D$.

Let $\sigma_1, \ldots, \sigma_n$ be n-ary predicate letters of D. We call $\sigma_1, \ldots, \sigma_n$ *pairwise disjoint* in ${}^{\nu}D$ iff for no $\alpha : \sigma_i(\alpha)$ in ${}^{\nu}D$ and $\sigma_j(\alpha)$ in ${}^{\nu}D$ unless $i = j$. (Concepts of formal representability, recursive enumerability, ν-inductive representability and ν-inductive enumerability are definable for definition systems.)

5. *Valuation forms, ν-valuation forms.* Henceforth we use the symbols $W,F,S_1,\dots,S_n$, $S'_2,\dots,S'_n$, $\overline{S'_2},\dots,\overline{S'_n}$, $R_2,\dots,R_n$ as *variables for predicate letters of definition systems.* These symbols should serve to suggest the meaning of the predicate letters for which they stand in the definition (e.g. W,F for predicate letters representing the predicates *true* and *false.* $S_1,\dots,S_n$ denote *sets of sentences*). We continue using D as a variable for definition systems and ν as a variable for elements of the class Z of constructive ordinals. The symbol $=$ is employed to indicate typographical identity.

A $(4n+1)$-tuple $\langle D,W,F,\nu,S_1,\dots,S_n,S'_2,\dots,S'_n,\overline{S'_2},\dots,\overline{S'_n},R_2,\dots,R_n\rangle$ of objects (typographic shapes) is called a *valuation form* iff the following conditions (1)-(3) are satisfied:

(1) $W,F,S_1,\dots,S_n$ are unary and $S'_2,\dots,S'_n$, $\overline{S'_2},\dots,\overline{S'_n}$, $R_2,\dots,R_n$ are binary predicate letters of definition systems. The predicate letters are all typographically distinct with the exception of those whose typographical identity is required by condition (2) below. D is a definition system whose stock of predicate letters begins with $W,F,S_1,\dots,S_n$, $S'_2,\dots,S'_n$, $\overline{S'_2},\dots,\overline{S'_n}$, $R_2,\dots,R_n$, $S_1,\dots,S_n$ are pairwise disjoint in 'D and for each $i(1\leq i\leq n)$ there are infinitely many α for which $S_i(\alpha)$ in 'D holds. For each $i(2\leq i\leq n)$ and any α such that $S_i(\alpha)$ in 'D there exists at least one β for which $S'_i(\beta,\alpha)$ in 'D holds, and for each pair α,β such that $S'_i(\beta,\alpha)$ in 'D: $S_i(\alpha)$ in 'D and $S_j(\beta)$ in 'D for some j $(1\leq j\leq n)$. All predicate letters in 'D are decidable in 'D.

(2) For each i $(2\leq i\leq n)$ D contains (a) an axiom $S'_i(x,y), F(x) \Longrightarrow F(y)$ or (b) an axiom $S'_i(x,y)$, $W(x) \Longrightarrow F(y)$ but not both.

In case (a) either (a_1) or (a_2) must hold:

(a_1) D contains an axiom Σ_1 such that for any α, if S_i

(α) in $'D$ and $\beta_{1\alpha},\ldots,\beta_{m\alpha}$ $(m>1)$ are all β for which S'_i, (β,α) in $'D$ holds, we may obtain a formula $W(\beta_{1\alpha}),\ldots,W(\beta_{m\alpha}) \Longrightarrow W(\alpha)$ as the result of substitution in Σ_1 in D, and further $\overline{S'_i}=S'_i$, $R_i=S'_i$.

(The class of all β with $S'_i(\beta,\alpha)$ in $'D$ may be called the *semantic base* of α in $'D$.)

(a₂) $\overline{S'_i}$ is the complement of S'_i in $'D$ and D contains the axioms $\overline{S'_i}(x,y) \Longrightarrow R_i(x,y)$, $W(x) \Longrightarrow R_i(x,y)$ and $\Lambda x\, R_i(x,y) \Longrightarrow W(y)$.

In case (b) either (b_1) or (b_2) must hold:

(b_1) D contains an axiom Σ_2 such that for any α, if $S_i(\alpha)$ in $'D$ and $\beta_{1\alpha},\ldots,\beta_{m\alpha}$ $(m>1)$ are all β for which S'_i, (β,α) in $'D$ holds, we may obtain a formula $F(\beta_{1\alpha}),\ldots,F(\beta_{m\alpha}) \Longrightarrow W(\alpha)$ as the result of substitution in Σ_2 (in D) and further $\overline{S'_i}=S'_i$, $R_i=S'_i$.

(b_2) $\overline{S'_i}$ is the complement of S'_i in $'D$ and D contains the axioms $\overline{S'_i}(x,y) \Longrightarrow R_i(x,y)$, $F(x) \Longrightarrow R_i(x,y)$ and $\Lambda x R_i(x,y) \Longrightarrow W(y)$.

(3) The predicate letters W,F,R_i $(1 \leq i \leq n)$ appear in D only in those axioms required by (2).

A definition system D is called a *ν-valuation form* iff there are $W,F,S_1,\ldots,S_n$, $S'_2,\ldots,S'_n$, $\overline{S'_2},\ldots,\overline{S'_n}$, $R_2,\ldots,R_n$ $(n\geq 2)$ such that $\langle D,W,F,\nu,S_1,\ldots,S_n, S'_2,\ldots,S'_n, \overline{S'_2},\ldots,\overline{S'_n}, R_2,\ldots,R_n \rangle$ is a valuation form.

A ν-valuation form D may be thought of as a system (list) of syntactic and semantic rules. The elements of the pairwise disjoint sets $S_1,\ldots,S_n$ are the *sentences*. For each sentence there is a *semantic base*. The semantic rules provide a criterion for evaluating each sentence based on the valuation of the elements of the semantic base of the sentence. A sentence is *positive in D* iff it belongs to the class S_i mentioned in condition (2a) of the definition of valuation form. A sentence is *negative in D* iff it is of type S_i mentioned in condition (2b). Conjunctions and universal quantifications would be examples of positive sentences in a valuation form for the syntax and semantics of classical predicate logic; negations would appear there as negative sentences. Disjunctions and existential quantifications would also be positive. The sole element of the semantic base of a disjunction $(\Phi \vee \Psi)$ would be the sentence $\neg(\neg\Phi \wedge \neg\Psi)$; the sole element of the semantic base of the existential sentence $\bigvee\alpha\Psi$ would be the sentence $\neg\bigwedge\alpha\neg\Psi$.

The stock of predicate letters $\gamma_1,\ldots,\gamma_p$ of a ν-valuation form determines the 4n predicate letters mentioned in the definition of the valuation form.

Though all concepts involved in a valuation form are decidable, we do not set $\nu=0$, because universal quantifiers may occasionally be useful in formulating decidable concepts. For similar practical reasons we distinguish cases (a_1) and (b_1) of condition (2): the use of universal formulas in the semantic rules for a sentence with a finite semantic base, like conjunction, would be superfluous. A ν-valuation form contains semantic rules for the assignment of truth values to sentences, but no sentence is actually given a truth value:

Theorem 1. Let D be a ν-valuation form whose predicate letters stock $\gamma_1,\ldots,\gamma_p$ begins with W,F. Then for no α $D\overset{\nu}{\vdash}W(\alpha)$ and for no α $D\overset{\nu}{\vdash}F(\alpha)$.

Proof: Condition (3) of the definition of ν-valuation forms requires that the predicate letters W,F (uniquely determined by the predicate letter stock) appearing in the ν-valuation form D occur only in the stipulated semantic rules, which are axioms of the form $\Phi \Longrightarrow \Psi$. For each ν, all ν-inductively provable formulas of D containing W or F are also formulas containing the symbol $\Longrightarrow$, because substitution allows only the derivation of arrow formulas from arrow formulas, modus ponens may not be applied to a sequence of arrow formulas and the premisses of the arrow formulas contain W,F which appear only in the arrow formulas mentioned.

6. *ν-valuations.* Let D and E be definition systems. E is an *extension* of D iff E is obtained by adding new axioms to D.

A n-ary predicate letter σ has in E the same extension as in D (relative to Z) means that for any n-tuple $\alpha_1,\ldots,\alpha_n$ of complex names $\sigma(\alpha_1,\ldots,\alpha_n)$ in ${}^{\nu}E$ if and only if $\sigma(\alpha_1,\ldots,\alpha_n)$ in ${}^{\nu}D$ (for any ν; $\nu \varepsilon Z$).

E is a ν-valuation over D means:

(1) D is a definition system and E is an extension of D.

(2) There is an n ($n \geq 2$) and predicate letters W,F, $S_1,\ldots,S_n$, $S'_2,\ldots,S'_n$, $\overline{S'_2},\ldots,\overline{S'_n}$, $R_2,\ldots,R_n$, such that $\langle D,W,F,\nu, S_1,\ldots,S_n, S'_2,\ldots,S'_n, \overline{S'_2},\ldots,\overline{S'_n}, R_2,\ldots,R_n \rangle$ is a valuation form, where every predicate letter except W,F,R_i has the same extension in E as in D relative to Z, R_i occurs only in such axioms of E which are axioms of D and for each $i(1 \leqslant i \leqslant n)$ holds: for each α such that $S_i(\alpha)$ in ${}^{\nu}D$ there is a $\mu<\nu$ or $\mu=O$ such that $W(\alpha)$ in ${}^{\mu}E$ or $F(\alpha)$ in ${}^{\mu}E$.

(3) For no α both $W(\alpha)$ in ${}^{\nu}E$ and $F(\alpha)$ in ${}^{\nu}E$.

Theorem 2 and Theorem 3 show that ν-valuations over definition systems have the properties one would expect on the basis of their semantic rules.

Theorem 2. Let D be a ν-valuation form with $W,F,S_1,\ldots,S_n$, $S'_2,\ldots,S'_n$, $\overline{S'_2},\ldots,\overline{S'_n}$, $R_2,\ldots,R_n$ as initial predicate letters of its predicate letter stock, and let E be a ν-valuation over D. For any positive α in D and some i ($1 \leq i \leq n$) assume $S_i(\alpha)$ in ${}^{\nu}E$: Then if for each β such that $S'_i(\beta,\alpha)$ in ${}^{\nu}E$ it is the case that $W(\beta)$ in ${}^{\nu}E$, then $W(\alpha)$ in ${}^{\nu}E$.

Proof: If the semantic base of an α is finite and the semantic rules for the corresponding sentence type contain no universal formula, then $W(\alpha)$ follows from the semantic rules with the same proofdegree ν as the proof of W (β_1) for the sentences $B_1, \ldots, \beta_m$ belonging to the semantic base of α $(1 \leq i \leq m)$. But if the semantic rules for α's sentence type contain a universal formula (which is necessarily the case, if the semantic base of α is infinite), then the valuation of α is guaranteed by means of the corresponding semantic rules because the elements of the semantic base are in this case semi-formally provable for some $\mu < \nu$. For similar reasons the following theorems hold, if D is a valuation form with $W, F, S_1, \ldots, S_n$, $S'_2, \ldots, S'_n$, $\overline{S'_2}, \ldots, \overline{S'_n}$, $R_2, \ldots, R_n$ as initial predicate letters of its predicate letter stock and E is a ν-valuation over D.

Theorem 3. For any negative α in D and some $i(1 \leq i \leq n)$ assume $S_i(\alpha)$ in 'E. Then if for each β such that $S'_i(\beta, \alpha)$ in 'E, it is the case that $F(\beta)$ in 'E, then $W(\alpha)$ in 'E.

Theorem 4. For any positive α in D and some $i(1 \leq i \leq n)$ assume $S_i(\alpha)$ in 'E. Then if there is a β such that $S'_i(\beta, \alpha)$ in 'E and $F(\beta)$ in 'E, then $F(\alpha)$ in 'E.

Theorem 5. For any negative α in D and some $i(1 \leq i \leq n)$ assume $S_i(\alpha)$ in 'E. Then if there is a β such that $S'_i(\beta, \alpha)$ in 'E and $W(\beta)$ in 'E, then $F(\alpha)$ in 'E.

The following theorems are consequences of the definition of the ν-valuation according to which there is no α for which both $W(\alpha)$ and $F(\alpha)$ hold in 'E.

Theorem 6. For any positive (negative) α in D and some $i(1 \leq i \leq n)$ assume $S_i(\alpha)$ in 'E and $W(\alpha)$. Then for any β: if $S'_i(\beta, \alpha)$ in 'E, then $W(\beta)$ $(F(\beta))$ in 'E.

Theorem 7. For any positive (negative) α in D and some

$i(1 \leq i \leq n)$ assume $S_i(\alpha)$ in $'E$ and $F(\alpha)$ in $'E$. Then there is a β such that $S_i'(\beta,\alpha)$ in $'E$ and $F(\beta)$ $(W(\beta))$ in $'E$.

A ν-valuation is by no means constructive in the sense that an assignment of truth values to certain sentences S_1 (prime sentences, atomic sentences) determines the truth value of all other sentences by means of the semantic (ascending) rules alone.

In order to make this intuitive notion of a *constructive valuation* precise, we first use ν-inductive proofs to define the concept of a basic assignment of truth values to the sentences of S_1.

7. *ν-basic assignments.* Let E and D be definition systems. We introduce w and f as further variables for predicate letters of definition systems.

E is a *ν-basic assignment over D* iff

(1) D is a ν-valuation form. E is an extension of D. The predicate letter stock of E ends with the two typographically distinct predicate letters w,f, which do not occur in D: W,F and S_1 are the initial predicate letters of the predicate letter stock of D. E contains the axioms $S_1(x), w(x) \Longrightarrow W(x)$ and $S_1(x), f(x) \Longrightarrow F(x)$. W,F and S_1 are the only predicate letters of D occurring in the axioms added to D in obtaining E; among the added axioms, the above - mentioned are the only ones containing the predicate letters W,F and S_1.

(2) If w,f are the last two predicate letters in the predicate letter stock of E, then for any α such that $S_1(\alpha)$ in $'D$ exactly one of the following holds: $w(\alpha)$ in $'E$ or $f(\alpha)$ in $'E$.

By (1) of the above definition, truth values can be assigned to sentences not belonging to S_1 only by applying the semantic rules of D to the given valuation of the sentences of S_1 (prime sentences, atomic sentences).

Condition (1) also guarantees that, with the sole exception of W and F, the extensions of all other predicate letters of D will remain the same in E.

8. *ν-constructive valuations.* E is a *ν-constructive valuation over D* iff E is a ν-basic assignment over D and E is a ν-valuation over D. The above definition of a constructive valuation delineates precisely the *intuitive notion of an assignment of truth values to sentences by rules.* This assignment *starts with a given valuation for the prime sentences (atomic sentences) and proceeds solely by the use of semantic rules all of which assign truth values only under certain conditions, never directly.* ν-valuations are not necessarily ν-constructive valuations. ν-valuations can adequately exhibit the semantics of an *impredicative language.*

9. *ν-semantics, ν-constructive semantics.* D is a *ν-semantic system* iff there is an E such that D is a ν-valuation form and E is a ν-valuation over D.

D is a *ν-constructive semantic system* iff there is an E such that D is a ν-valuation form and E is a ν-constructive valuation over D.

D is a *non-constructive or impredicative semantic system* (relative to Z) iff for some ν D is a ν-semantic system, but for no ν ($\nu \varepsilon Z$) is D a ν-constructive semantic system. Constructive semantic systems obviously exist for classical propositional logic, classical predicate logic and the ramified theory of types. For predicate logic it is sufficient to consider a class of constructive ordinals up to ω. For the ramified theory of types a more comprehensive class Z would be needed according to the order of the sentences. Simple type theory may be used to illustrate the difference between ν-semantics and ν-constructive semantics. For this theory there is a ν-semantic system ($\nu = \omega$) but no ν provides a ν-constructive semantic system.

Let D be a ν-semantic system whose predicate letter stock begins with $W, F, S_1, \ldots, S_n$. We then call the set A of α such that for some $i(1 \leq i \leq n)$ $S_i(\alpha)$ in $^{\nu}D$ a *ν-language*; if D is ν-constructive, we call A a *ν-constructive language.*

The set of sentences of simple type theory provides an example of a *non-constructive* (impredicative) *language*, i.e., a set S of sentences for which there is no ν such that S is a ν-constructive language.

10. *ν-logical truth, ν-constructive logical truth.* α is *ν-logically true* in D iff D is a ν-semantic system whose predicate letter stock begins with W and for every ν-valuation E over D: $E \overset{\nu}{\vdash} W(\alpha)$.

α is *constructively ν-logically true* iff D is a ν-constructive semantic system and for every ν-constructive valuation E over D: $E \overset{\nu}{\vdash} W(\alpha)$, W being the first predicate letter in the stock of D. We could similarly define concepts of ν-logical falsity, ν-refutability, ν-satisfiability, ν-model, etc., for example, call α *constructively ν-logically satisfiable* iff D is a ν-constructive semantic system and there is a ν-constructive valuation E over D such that $E \overset{\nu}{\vdash} W(\alpha)$, where W is the first predicate letter of the predicate letter stock of D (E is then called a *ν-constructive model* for α). All ν-concepts are relative to the chosen set Z of constructive ordinals ($\nu \varepsilon Z$). Thus, the fact that a definition system is not a ν-semantic system for all ν of a certain class Z does not prevent it from being a μ-semantic system, or even a μ-constructive semantic system, for some μ ($\nu < \mu$) of a more comprehensive class Z′, and the same can be said of other concepts like ν-language or ν-logical truth.

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RESUMEN

Se definen expansiones de sistemas de Post ([1]-[3]) y se usan para construir sistemas semánticos en los que se permite la cuantificación sobre individuos pero no sobre clases. El metalenguaje empleado para definir el concepto de verdad lógica para el lenguaje objeto es un lenguaje de primer orden mientras que los lenguajes objeto son lenguajes de un orden superior arbitrario.

La construcción de tales sistemas semánticos muestra que incluso los lenguajes platónicos pueden tener un fundamento no platónico. Los sistemas semánticos no se definen para lenguajes específicos; en lugar de esto, definimos un concepto general de lenguaje (lenguaje ν) relativo a una clase Z de ordinales construibles ($\nu \varepsilon Z$). Ejemplos de tales lenguajes lo son los lenguajes usuales de la lógica clásica de primer orden, así como los de las teorías simple y ramificada de los tipos.

(Traducción de José A. Robles)