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Mark Colyvan, *An Introduction to the Philosophy of Mathematics*, Cambridge University Press, Cambridge, 2012, 188 pp.

Colyvan's *Introduction* presents an up-to-date, clearly written journey through the recent debates on the philosophy of mathematics. It offers a valuable examination of some of the current main problems, fulfilling the primary goals of an introductory reading for those working in the discipline. In my judgment, however, it goes beyond, addressing some of the most interesting issues in the philosophy of mathematics and their relevance for the philosophy of science.

Some of the themes explored throughout its pages are these: a chart of the debates on the philosophy of mathematics (Ch. 1); the limits of mathematics (Ch. 2); realist philosophies of mathematics (Ch. 3); anti-realist philosophies of mathematics (Ch. 4); mathematical explanation (Ch. 5); the applicability of mathematics (Ch. 6); inconsistent mathematics (Ch. 7); mathematical notation (Ch. 8); and the examination of twenty philosophically interesting mathematical theorems (Ch. 9). In order to offer a fairly comprehensive review of the volume, in what follows I firstly examine the debate between realist and anti-realist philosophies of mathematics and secondly I briefly focus on the nature of mathematical explanation, the applicability of mathematics, and the viability of the mapping account.

## I

According to Colyvan, Benacerraf's (1983a; 1983b) landmark papers and the indispensability argument, independently elaborated by Quine (1976; 1981) and Putnam (1979), chart the course through the problems in the philosophy of mathematics over the last decades. On a first approach, these problems can be classified into three, namely: first, the elaboration of a uniform semantics for both mathematical and non-mathematical discourse; second, an epistemology of mathematics that satisfactorily addresses the question of how we come to know mathematical entities in the first place; and third, ontological problems related to the nature of mathematical entities. Colyvan makes some interesting comparisons between realist and anti-realist philosophies of mathematics and their attempts to answer these questions.

Following Putnam (1979, p. 70), the author distinguishes realism as a thesis about the objectivity of mathematical knowledge, on one hand, and as a thesis about the existence of mathematical entities, on the other (pp. 36–38). From a realist perspective, mathematical statements can be regarded as objectively true if there are some mathematical posits that make them true. Once realists have agreed on this, viz., that mathematical existence is a precondition of mathematical knowledge, a new source of possible disagreement emerges from the consideration of the nature of the mathematical entities that supposedly exist. At this point, realism is put forward in different shapes and flavours. For instance, a first form of realism is full-blooded Platonism, which endorses the view that mathematical entities are abstract in nature, having neither spatio-temporal location nor causal powers. A second form is physicalist mathematical realism, which entertains the idea that mathematical entities are physical in nature, belonging to the spatio-temporal causal link (Bigelow 1988; Maddy 1990). And a third form is the structuralist view, which advocates that the subject matter of mathematics is the structural relation, instantiated or not, among mathematical entities (Resnik 1997, pp. 36–41).

Let us look in some further detail at the full-blooded Platonistic version of mathematical realism. Its central tenet can be formulated as follows: every consistent mathematical theory truly describes some part of the mathematical universe (p. 38). Full-blooded Platonism meets the semantic and ontological requirements by postulating a very rich ontology, since it is claimed that statements such as "5 + 7 = 12" and "the atomic number of gold is 72" are both true if we accept that there actually are things such as numbers and chemical properties. Nonetheless, it counts against this view that it does not straightforwardly offer a response to the epistemic problem, viz., how we come to know non-causal, non-spatio-temporally located, abstract mathematical entities in the first place.

Arguments for realism in the philosophy of mathematics usually appeal to some form of indispensability argument (pp. 41–54). Colyvan's previous contribution (2001a, pp. 6–17) may well be considered the best treatment of its kind. As outlined in the present volume, the general form of the argument is as follows:

(P1) We ought to have ontological commitment to all and only the entities that are indispensable to our current best scientific theories.

(P2) Mathematical entities are indispensable to our best scientific theories.

(C) We ought to have ontological commitment to mathematical entities (p. 43).

*Prima facie*, (P1) relies on the assumption that scientific theories are our best guide to the furniture of reality, whilst (P2) puts forward the idea that in fact scientific theorising quantifies over both mathematical and physical entities. *Ergo*, (C) concludes that we should accept the existence of mathematical entities just as we typically accept, from a standard scientific realist perspective, the existence of theoretical entities. Furthermore, the argument advocates confirmational holism, which broadly speaking is the view that "theories are confirmed or disconfirmed as wholes" (p. 45). Thus, granted confirmational holism, it can be claimed that the mathematical component of scientific theories is confirmed along with the confirmation of the physical component of the same scientific theories.

So far so good for mathematical realism. However, several proposals of mathematical anti-realism make a compelling case against indispensability arguments. In this respect, Colyvan examines the works of Field (1980; 1989), Maddy (1990), Azzouni (2004), and Yablo (1998). I shall briefly look into the first two.

Field's fictionalism argues against (P2). In my view, he elegantly manages to meet the semantic, epistemic, and ontological problems in a very attractive manner. Regarding ontology, he contends that there are no such things as mathematical entities or, if there are, they are mere fictions belonging to the story of mathematics. This allows Field to meet both the epistemological and semantic challenges. On one hand, as to the former concern, he defends the claim that there is no mathematical knowledge apart from the knowledge of the fiction of mathematics. On the other, as to the latter concern, he claims that we should read the statement "2 is larger than 4" at face value just like the statement "Sydney is larger than Amsterdam". Here, however, we should deny truth to the first statement, but not to the second, given that mathematical entities do not exist, unlike the aforementioned cities (pp. 56–57).

Yet Field's nominalisation program is further developed in two more respects, viz., dispensability and conservativeness. In my view, they contribute in their own ways to debunking the realist's reliance on indispensability arguments. Let us briefly summarise these arguments. The first seeks to show that mathematical entities are dispensable after all. Field makes a case for his idea by constructing part of the Newtonian gravitational theory without quantifying over mathematical items (pp. 59–60). This is, according to Colyvan, a particular approach to the development of a nominalistic philosophy of mathematics that partially exemplifies what he calls *the hard road* *to nominalism*,[1] which, if generalised, represents the endeavour of constructing all of science without quantifying over mathematical entities. The author explicitly notices that Field does not advocate doing science without mathematics, but spends a good deal of effort in showing how science *can be done* without it. To some extent, this suffices to show that, at least in the case examined by Field, mathematical entities are dispensable. Regarding the conservativeness argument, it proposes that mathematics is conservative in scientific theorising in the following terms, namely: "a mathematical theory, when combined with any nominalistic scientific theory, does not yield nominalistic consequences that could not have been derived from the nominalistic theory alone" (pp. 60–61). That is to say, the mathematical component of a scientific theory does not add any surplus nominalistic content to the theory as a whole. Were the mathematical component eliminated, the nominalistic content of a theory would remain the same.

On a different approach, Maddy, although well known as a realist about mathematics, objects to (P1) maintaining that we should not have ontological commitment to all the entities posited by scientific theories. Considering the case of how the concept of atom became a core element in contemporary physics, Maddy looks into the epistemic attitudes that scientists actually profess toward the components of well-confirmed scientific theories. These attitudes, indeed, happen to vary from belief, through tolerance, to outright rejection. Thus, even though the concept of atom was a fundamental unit of chemistry around 1860, it was universally accepted only at the beginning of the twentieth century after the work of Einstein and Perrin on Brownian motion (pp. 48–49). In the same vein, Maddy stresses that the mathematical component of theories falls within the idealised elements of scientific theories, and not the true ones. Therefore, philosophers should not take the supposed indispensability of mathematics to be a reliable guide to the truth of mathematics or to the truth of scientific theories.

Lastly, appealing to actual mathematical practice as well, Maddy rejects a possible methodological application of confirmational holism to mathematics, mainly because to do so would restrict our acceptance of mathematics only to those aspects of mathematical theorising that fit within the scientific web of belief. This is actually not the case in the intertwining of current mathematical and scientific practice. For instance, mathematicians can be genuinely engaged in some tasks

[1] See also Colyvan (2006).

that can hardly pass the test of philosophers' confirmational holism. It may well be the case that mathematical theorising is far from cohering with the web of scientific belief, without this being detrimental for mathematics' practitioners. Thus, the dilemma is this: either we embrace philosophers' belief in confirmational holism, or we look, instead, at mathematics as currently practiced by mathematicians. Maddy recommends the latter, and the example from Maddy that Colyvan chose to examine is the settlement of independent questions, such as the truth of the continuum hypothesis. Quite likely, this problem would not have been correctly addressed if the new axioms proposed to complement ZFC (the Zermelo-Fraenkel set theory with the axiom of choice, avoiding the paradoxes of naïve set theory and Russell's paradox) had been assessed according to whether or not they cohered with the best scientific theories available at the time. Set theorists, in fact, need not think of coherence with scientific theories as a criterion to be considered when it comes to problems such as the truth of the continuum hypothesis (p. 50).

## II

Moving forward, the following three questions require an independent treatment: first, mathematical explanation; second, the applicability of mathematics; and third, the viability of the mapping account. I address them below outlining at some relevant points their interrelationship.

Let us begin with mathematical explanation. Rather than accommodating mathematical explanation within traditional philosophical categories, Colyvan proposes to distinguish between intra- and extra-mathematical explanations. The former are cases of mathematical explanations of mathematical facts, such as explanatory proofs of theorems, while the latter are cases of mathematical explanations of physical facts, as can be found in mathematical formulations of scientific theories, principles, and laws, for instance (pp. 75 ff.).

Let us look at extra-mathematical explanations in particular, which are naturally intertwined with issues related to the applicability of mathematics. Considering this from a general perspective, the history of modern science gives us a good deal of awe-inspiring evidence of extra-mathematical explanations, which illustrates the successful application of mathematics to the natural sciences in searching for objective features of reality. In fact, in some areas of science explanation is largely possible thanks to sophisticated mathematical notation and operation rules. Furthermore, a priori mathematical theorising and its application to the physical realm have clearly triggered the formulation of scientific theories, construction of scientific representation, and progress in scientific discoveries overall. In order to exemplify these points in some detail, Colyvan pays particular attention to the cases of the hexagonal structure of the hive-bee honeycomb, the asteroid belt, the Lorentz contractions (pp. 90–93), and Maxwell's electro-magnetic theory (pp. 101–102). In the same vein, philosophers of mathematics have done substantial work extending this line of analysis to other parts of high-theoretical physics and scientific cosmology, where mathematics appears to be the best grasp that we have of some physical systems at certain micro and macro levels.[2]

Regarding the second question, i.e., the applicability of mathematics, scientists, philosophers, and mathematicians have wondered about the *unreasonable* character of its application and its enormous epistemic contribution to science. To my knowledge, Steiner's (1998), Pincock's (2012), and Bangu's (2012) contributions offer the best and most comprehensive accounts of the applicability of mathematics. According to Colyvan (pp. 98–100), the *locus classicus* in this respect is a passage from Wigner in which he speaks of the "miracle of the appropriateness of the language of mathematics" as a "wonderful gift" (1960, p. 14). The literature goes further, Colyvan claims. Hertz, for instance, as quoted by Dyson (1962, p. 129), emphasises the "feeling" that mathematical formulae "have independent existence and intelligence of their own." Weinberg, on the other hand, highlights "mathematical beauty" (1993, p. 125) as an epistemic guide to the applicability of mathematics, whilst Steiner thinks of mathematicians as "closer to the artist than the explorer", considering the enormous creativity needed to apply high-mathematics to physical systems (1995, p. 154).[3]

[2] See, for instance, Lyon and Colyvan (2008) and Batterman (2010).

[3] The referee for *Critica* claims: "the quotes from Hertz and Weinberg do not seem to be about the application of mathematics or mathematical explanation" and that "they have nothing to do with external mathematical explanation". I take her observation to suggest a genuine philosophical problem regarding the interpretation of Hertz's and Weinberg's claims and I think she is to a great extent correct, since both Weinberg and Herz —and Colyvan, if I am understanding his views correctly— are dealing here with a different, more general twofold issue, viz.: on one hand, the applicability of mathematics to physical phenomena and, on the other, the epistemic contribution of mathematics to science in uncovering objective features of reality. Their claims —and the current issue at stake here— are not about intra- or extra-mathematical explanations in particular, but only about the general fact of how the applicability of mathematics and its epistemic contribution to science *appear to be unreasonable*, in the sense of being at first sight beyond any reasonable explanation.

Nonetheless, a sensible naturalistic stance restrains us from pursuing this way of philosophical speculation. There must be something else; that is to say, it cannot only be wonders, miracles, feelings, and the like. Rather, it is a proper task of the philosopher of mathematics to explain the epistemic success and general applicability of mathematics so as to make it reasonable. Taking this challenge seriously, Colyvan's analysis sheds light on the issues, outlining several considerations that contribute to our understanding of the applicability of mathematics. The first consideration regards difficulties of mathematical modelling when it comes to getting "the mathematics and the world to see eye to eye" (p. 104). The second concerns the fact that we do not have a very good grip on what counts as unreasonable, because, for instance, we would need to know exhaustively not only how many mathematical projects have succeeded, but also how many have failed through the history of mathematics. And the third points out that mathematics has become the best tool that we currently have for modelling different parcels of reality, to such an extent that we can expect science to deliver a physico-mathematical picture of reality, like the one that we have gotten thus far (pp. 104–105).

At this point, it is still necessary to offer a detailed account of the applicability in question. Let us look then at our third issue, namely, the mapping approach, which has been differently formulated in the relevant literature (pp. 106–115). The mapping approach examined by Colyvan attempts to offer an account of both mathematical explanation and the applicability of mathematics in terms of construction of mathematical models. From this viewpoint, mathematical models work like maps: they represent salient features of their target systems by similarity-based relations. The fact that a particular model is intended to pick up only salient characteristics of their respective physical systems does not mean to imply that mathematical models are to be interpreted as mere simplifications of what is really going on out there. Rather, the mapping approach considers mathematics as "a source of structures for scientific theorising" (p. 108) in the following sense: mathematics provides different sorts of highly theoretical abstract structures, which can afterwards be employed by scientists who are in search of the nature and structure of reality. In sum, the mapping account proposes that mathematics ideally refers to reality

Looking at the applicability question from this perspective makes it worth noting mathematicians' and scientists' expressions of wonder triggered by the in some cases unexpected applicability of mathematics. Due to the restrictions of space proper to a review, I cannot go into the details of this argument now. See for further analysis Colyvan (2001b).

by similarity-based relations between particular mathematical models and their respective target systems.

Colyvan's meticulous examination of mathematical models of population employed in ecology to predict the abundance of real population of some organisms is a very illustrative instance of the mapping account at work (pp. 110–115). According to the mapping approach, the examination of how the construction and application of models work shows that a mathematical model is explanatory, viz., it has successfully mapped its target system, in so far as, first, it delivers essentially physical explanations of physical facts; second, it abstracts the salient features of the target system; and third, it can lead us to new ideas, sometimes predictions and discoveries, by analogical reasoning.

I can recommend Colyvan's new contribution as an accurate and accessible preamble to some of the most interesting riddles in the discipline. The book offers a great deal more than the arguments outlined above. Not only philosophers, but also mathematicians and scientists, can benefit from it as a thoroughly informed, philosophically insightful guide to debates that encompass their areas in the pursuit of a better understanding of reality.[4]

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## REFERENCES

Azzouni, J., 2004, *Deflating Existential Consequence: A Case for Nominalism*, Oxford University Press, Oxford.

Bangu, S., 2012, *The Applicability of Mathematics in Science: Indispensability and Ontology*, Palgrave Macmillan, London.

Batterman, R., 2010, "On the Explanatory Role of Mathematics in Empirical Science", *British Journal for the Philosophy of Science*, vol. 61, no. 1, pp. 1–25.

Benacerraf, P., 1983a, "What Numbers Could Not Be", in Benacerraf and Putnam 1983, pp. 272–294.

———, 1983b, "Mathematical Truth", in Benacerraf and Putnam 1983, pp. 403–420.

Benacerraf, P. and H. Putnam (eds.), 1983, *Philosophy of Mathematics: Selected Readings*, Cambridge University Press, Cambridge.

Bigelow, J., 1988, *The Reality of Numbers: A Physicalist's Philosophy of Mathematics*, Clarendon Press, Oxford.

Cheyne, C. and J. Worral (eds.), 2006, *Rationalilty and Reality: Conversations with Alan Musgrave*, Springer, Dordrecht.

Colyvan, M. 2006, "Scientific Realism and Mathematical Nominalism: A Marriage Made in Hell", en Cheyne and Worral 2006, pp. 225–237.

———, 2001a, *The Indispensability Argument*, Oxford University Press, Oxford.

———, 2001b, "The Miracle of Applied Mathematics", *Synthese*, vol. 127, no. 3, pp. 265–277.

Dyson, F., 1962, "Mathematics in the Physical Sciences", *Scientific American*, vol. 211, no. 3, pp. 128–146.

Field, H., 1989, *Realism, Mathematics, and Modality*, Blackwell, Oxford.

———, 1980, *Science without Numbers: A Defence of Nominalism*, Blackwell, Oxford.

Lyon, A. and M. Colyvan, 2008, "The Explanatory Power of Phase Spaces", *Philosophia Mathematica*, vol. 3, no. 16, pp. 227–243.

Maddy, P., 1990, *Realism in Mathematics*, Clarendon Press, Oxford.

Pincock, C., 2012, *Mathematics and Scientific Representation*, Oxford University Press, Oxford.

Putnam, H., 1979, *Mathematics, Matter, and Method: Philosophical Papers Volume 1*, Cambridge University Press, Cambridge.

Resnik, M.D., 1997, *Mathematics as a Science of Patterns*, Clarendon Press, Oxford.

Quine, W.V.O., 1981, *Theories and Things*, Harvard University Press, Cambridge, Mass.

———, 1976, *The Ways of Paradox and Other Essays*, Harvard University Press, Cambridge, Mass.

Steiner, M., 1998, *The Applicability of Mathematics as a Philosophical Problem*, Harvard University Press, Cambridge, Mass.

Steiner, M., 1995, "The Applicabilities of Mathematics", *Philosophia Mathematica*, vol. 3, pp. 129–156.
Weinberg, S., 1993, *Dreams of a Final Theory*, Vintage, London.
Wigner, E., 1960, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences", *Communications on Pure and Applied Mathematics*, vol. 13, pp. 1–14.
Yablo, S., 1998, "Does Ontology Rest on a Mistake", *Proceedings of the Aristotelian Society*, Supplementary Volumes, vol. 72, pp. 229–261.