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PLUS AND MINUS

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Is it possible to say, in a clear and precise way, just what constitutes the distinction between logical expressions (formatives, syncategoremata, particles, constants) and nonlogical expressions (material expressions, categoremata, terms, variables)? Pessimism concerning this seems to be the rule among most contemporary logicians. For example, Tarski: "no objective grounds are known to me which permit us to draw a sharp boundary between [logical and extralogical] terms" ([30], pp. 418–419); Mates: "unfortunately the question as to which words should be considered logical and which not involves a certain amount of arbitrariness" ([18], p. 14); Quine: "Each such word is in a class fairly nearly by itself; few words are interchangeable with it *salva congruitate*. Instead of listing a construction applicable to such a word and to few if any others, we simply count the word an integral part of the construction itself. Such is the status of particles" ([20], p. 29); Allwood, Anderson and Dahl: "In the last instance it is a matter of decision whether a word belongs to the logical vocabulary or not" ([1], p. 24).

In the face of today's overwhelming pessimism we must remind ourselves that there have been earlier times of optimism. For example, Leibniz: "So just as there are two primary signs of algebra and analytics, + and −, in the same way there are, as it were, two copulas 'is' and 'is not' " ([16], p. 3); DeMorgan:

"I think it is reasonably probable that the advance of symbolic logic will lead to a calculus of opposite relations, for mere inference, as general as that of + and − in algebra" ([2], p. 26); Sommers: "all formatives—including propositional 'constants'—are analogous to plus and minus signs of arithmetic"([25], p. 249).

If these optimists are correct, then not only it is possible to draw a clear and precise distinction between the logical and the extralogical expressions of a language, it is also possible to give a very simple characterization of the nature of logical expressions—they are all signs of opposition, analogous to the oppositional signs of mathematics. Such prospects are surely attractive. If so, however, why have most contemporary logicians turned pessimistic? The answer, I think, lies in the shift from a traditional account of logical syntax to a Fregean account, rather than dwell on the story of that shift. We want to look closely at the consequences of the traditionalists' optimism with regard to the logical/extralogical distinction and the nature of logical expressions. Any substantiation of the attractive prospects offered by the traditionalist view must inevitably cast some doubt on the Fregean view and its concomitant pessimism.

1. Let us begin by looking at the signs of opposition used in mathematics—the plus and minus signs. Such signs are systematically ambiguous. Consider the expression

(1) $$-(2 + 3)$$

Here the minus sign is a unary formative. The expression in parentheses represents a number and the minus sign represents its negation. Thus it is equivalent to '-5', an explicitly negative number. But we could distribute the minus sign into the parenthetical expression. Thus:

(2) $$-2 - 3$$

In this case the first minus is clearly a unary formative. But what of the second one? We could be adding two negative numbers, i.e.

(3) $$(-2) + (-3)$$

or we could be subtracting positive 3 from negative 2, i.e.

(4) $$(-2) - (+3)$$

in which case the second minus sign is not a unary formative but binary. It marks the operation of subtraction on an ordered pair of numbers. Indeed, this ambiguity between a unary and binary use characterizes the plus sign as well. The parenthetical expression in (1) must be viewed as the addition (a binary operation) of two positive (a unary operation) numbers, i.e.

(1.1) $$-((+2) + (+3))$$

Mathematicians tend to omit plus signs whenever convenient (taking unmarked expressions to be implicitly positive). So the plus on 2 is omitted. But which of the next two pluses is omitted? The first of these is the binary formative marking addition. The second is the unary formative used to make 3 positive. One is omitted—and it is clearly the unary plus. Indeed, the convention here is quite uniform and universal: unary plus signs may be suppressed; binary signs, plus or minus, are never suppressed. That is why (4) is normally written as (2).

Notice that it is possible to do arithmetic, say, using only a negative unary formative and a positive binary one. In so doing one simply adds; subtraction is replaced by the addition of negative numbers. In such a system of arithmetic we could only introduce a binary negation (subtraction) by defining it in terms of numerical negation and addition (our "basic" unary and binary formatives respectively). We might think of a system using only such primitive expressions as itself *basic*; one which admits defined expressions (e.g. binary negation in our example) would be an *amplified* system.

2. The systematic ambiguity of plus and minus expressions in mathematical language is not only benign, it is a source of great expressive power for the mathematician. Leibniz, DeMorgan and Sommers have suggested that natural language has a logic which, like mathematics, makes use of two kinds of basic expressions, the signs of opposition. In fact, their common position seems to be that all the expressions of natural language which carry the responsibility for determining logical form (viz. constants) are either positive signs or negative signs or signs definable in terms of these.

Now if this is so it means, among other things, that one could build an artificial formal language which would model natural language by using the mathematician's opposition signs for all formatives. The result would be an algorithm for natural language which would model natural statements as arithmetical formulae and inference as arithmetical calculation. There is little doubt that this was Leibniz's goal throughout his logical studies. And Sommers has come very close to reaching that goal in his own logical work (see [25]–[28]). It would seem, therefore, that the idea of using signs of opposition to model natural language formatives is a good one, leading, as it seems to have done, to rich programs of logical investigation and to viable systems for logical reckoning.

One of the consequences of this idea has been, as we saw earlier, great optimism among those who have shared this idea that a clear and precise account of the nature of logical formatives, and their distinction from nonlogical expressions, can be provided. In a sense, their account is quite simple: logical formatives, unlike other expressions, are oppositional in just the way that plus and minus are oppositional in mathematics. But to fully appreciate this kind of account we need to look more closely at the oppositional character of formatives, their role in inferences, and the kind of algorithm which could model those inferences. Of course, the best way to achieve a full understanding of such an idea and all of its consequences for logic would be a close inspection of Sommers' completion of the Leibnizian program.

3. Let us imagine a *basic* language. Its lexicon, or vocabulary, consists of *terms* (we will say that all these terms are English words and phrases). A sentence in the basic language is formed by using terms and formatives. There are two formatives. One formative simply applies to a term and results in another term. It is 'non'. Applying 'non' to a term results in a new term which is the *negative* of the original *term*. Note that 'non', the negation expression, is a unary formative; it applies to terms one at a time. The second, and only other, formative applies to terms two at a time; it is binary. The binary formative is 'and'. Applying 'and' to a pair of terms results in a *phrase*. Every phrase is also a term (thus every term is either a *simple* term or a phrase). The 'and' has an important formal feature which will be exploited throughout the language—it is *symmetric*.

Before continuing it is important to point out that the basic language is not English, or even a fragment of English (though ideally it will share with English just those logical characteristics in which we are interested). This means that 'non' and 'and' are not the English words 'non' and 'and'. Were we to attempt to translate from English into the basic language we would translate many expressions as 'non' and a great many as 'and'. Here are some important examples.

Non	*And*
not	both... and
it is not the case that	belongs to some
un	is true of some
dis	is true of at least one
less	is true of a
it is false that	belongs to a
non	belongs to at least one
	is
	are

Non	*And*
	were
	is the same as
	is identical to
	but
	and

English expressions on such a list are not necessarily synonymous. But they do all share the same formal features. In the case of 'and' for example all the English expressions are, like 'and', symmetric binary formatives.

Among the phrases formed by use of the binary formative are *sentences*. Whether a phrase is a sentence or not depends upon how the formative is interpreted. So it is a matter of semantics and pragmatics rather than syntax. In a natural language like English we distinguish between those binary formatives which result in sentences and those which result in nonsentential phrases. For example, 'is' forms sentences from pairs of terms ('Man is mortal'), while 'and'—the English 'and' now—forms nonsentential phrases from simple terms or other nonsentential phrases ('man and beast', 'man and mortal but happy') and sentences from sentences ('Man is mortal and art is immortal'). We will often call nonsentential phrases *compound terms*. Nevertheless, from our purely formal point of view, all such binary formatives are translatable into the basic language as 'and', since all share the same formal features.

Our basic language is quite simple in that its grammar is exceedingly spare. Any term can be negated. Any pair of terms can be connected by 'and' to form a phrase. Any phrase is itself a term (thus it can be negated and it can be connected by 'and' to any other term). This syntactical simplicity—along with our choice of 'non' and 'and' as our only formative expressions suggests a simple algebraic algorithm for modelling expressions in our basic language. Let us symbolize (model) each simple term by an uppercase letter. Let us also symbolize 'non' by a minus sign and 'and' by a plus sign. Such a plus-minus algebra, where plus is binary and minus is unary, will be a model of our basic language. Ideally then, algebraic manipulation of expressions in the algebra will model the behavior of expressions in our basic language. This will be so as long as we remember that the plus in our algebra is a symmetric binary formative and the minus is a unary formative.

Here are the formation rules (rules of syntax, grammar) for the algebra.

(i) Any letter is a term

(ii) The result of prefixing a minus to a term is a (negated) term

(iii) The result of placing a plus between two terms is a term (phrase)

(iv) Any phrase is a term

In producing phrases one or more of whose terms are themselves phrases we will make use of various kinds of parentheses in the usual way to resolve syntactical amibiguities.

We saw earlier that 'and' (and now '+') is symmetric. A second formal feature which it has is associativity. We exploit these features in formulating two important transformation rules (rules of derivation).

> Commutation: Any phrase of the form '$X+Y$' is equivalent to a phrase of the form '$Y + X$'
>
> Association: Any phrase of the form '$(X+Y)+Z$' is equivalent to a phrase of the form '$X + (Y + Z)$'

Whenever we are given a sentence (sentential phrase, phrase interpreted as a sentence) or a pair of sentences it is possible to derive a new sentence. In doing so we make use of Commutation or Association or some other rule of derivation. Our formulation of Commutation and Association involved the formation of phrases from terms (using plus). But often derivation procedes while inattentive to the internal structures of the sentences involved. Let us institute a method for symbolizing sentences, then, which allows us, when we need to, to ignore their internal syntax. We will use lowercase letters to model sentences (keeping in mind however that every sentence is nonetheless a term). An important rule of derivation involving sentences is

> Simplification: Any sentence of the form 'p' is derivable from a sentence of the form '$p + q$'

A sentence, e.g. '$p + q$', which is formed from two other sentences by use of the binary formative is called a *conjunctive sentence* (or simply a conjunction). A nonsentential phrase, e.g. '$A + B$', which is formed from two other terms by use of the binary formative is called a *conjunctive phrase*. Simplification applies only to conjunctive phrases which are sentential, i.e., to conjunctive sentences.

Our algorithm has one further rule of derivation. This one involves the unary formative. The new rule is

> Double Negation: Any term of the form '$- - X$' is equivalent to a term of the form 'X'

4. Much of English can be translated into our basic language and can then be modeled by our symbolism. The algorithm can then be used to model inferences. Consider the following ordinary English sentences.

(a) Some senators are honest

(b) Some politicians are liars and frauds

(c) Some politicians are not liars

(d) A senator was dènounced

(e) Some senator was not honest

(f) It is false that a senator was bribed

(g) An honest but kind man is unloved

(h) Cicero is a senator

(i) Cicero and Marcus are allies

(j) Cicero is Tully

(k) Honesty belongs to at least one senator

We can translate these into sentences of our basic language and then symbolize them as follows.

(a.1) $S + H$

(b.1) $P + (L + F)$

(c.1) $P + -L$

(d.1) $S + D$

(e.1) $S + -H$

(f.1) $-(S + B)$

(g.1) $((H + K) + M) + -L$

(h.1) $C + S$

(i.1) $(C + M) + A$

(j.1) $C + T$

(k.1) $H + S$

Inference is modeled by applying our derivation rules to such formulae. For example, from (g.1) we could derive the formula for 'Some thing which is unloved is a man'. Thus:

1. $((H + K) + M) + -L$	premise
2. $(H + K) + (M + -L)$	1. Association
3. $(M + -L) + (H + K)$	2. Commutation
4. $M + -L$	3. Simplification
5. $-L + M$	4. Commutation

However, in time, inference-modeling will reveal a sever limitation on the expressive powers of our new language. Consider (a) and (k) above. By Commutation they should be equivalent, each derivable from the other. Thus, '$S + H$'='$H + S$'. But suppose from (a) I wish to derive 'Something honest is a senator'. This sentence is not synonymous with (k), yet it, like (k), must be formulated as (k.1), i.e. '$H + S$'. A simple way to avoid such a limitation (one suggested by Aristotle's custom in the *Analytics*) is to require that all basic sentences of the form 'Some *A* is/are/was/were/etc. *B*' be paraphrased before formalization as sentences of the form '*B* belongs to some *A*'. This would then mean that any formula of the form '$B + A$' would have to be read as '*B* belongs to some *A*', and it could *not* be read as 'Some *B* is *A*'.

Let us assume that we could resolve any other such problems by adopting similar conventions of symbolization. Still our basic language, in spite of some increase in its expressive power now, is, by comparison to any natural language, such as English, still quite limited in its powers of expression. Is it possible to expand or modify the language so that its expressive capacities more closely approach those of a natural language?

5. Let us begin by demanding that any statement (sentence used to make a statement, express a proposition) of English to be formulated into the language of our symbolic algorithm first be paraphrased canonically. The canonical fragment of English, then, will consist of (a) basic sentences having one of the following forms: '*A* belongs to some *B*', 'non*A* belongs to some *B*', 'Non: *A* belongs to some *B*' or 'Non: non*A* belongs to some *B*'; and (b) *amplified* sentences. Amplified sentences will contain one or more defined formative expressions. Any defined formative will be defined in terms of just our unary 'non' and binary 'and'.

Consider the quite unexceptional English sentence 'Every dog is a canine'. It is not a basic canonical sentence as it stands.

But it could be paraphrased by a sentence which, though less colloquial, is a basic canonical sentence, viz.

Every dog is a canine

No dog is not a canine

It is not the case that some dog is not a canine

It is not the case that some dog is a noncanine

Non: some dog is a noncanine

Non: noncanine belongs to some dog

This final paraphrase does have one of our four basic canonical forms. Though paraphrasing of this sort cannot be completely eliminated, it is possible to minimize it by the introduction of defined formatives. In the above example, for instance, it would be easier and more natural to have a straightforward way to express the English formative expression 'every. . . is. . . ' How might this be done?

Let us symbolize our four basic canonical forms of statements.

English	*Canonical Form*	*Symbolization*
Some boy is able	Ability belongs to some boy	$A + B$
Some boy is unable	Nonability belongs to some boy	$-A + B$
No boy is able	Non: ability belongs to some boy	$-(A + B)$
No boy is unable	Non: nonability belongs to some boy	$-(-A + B)$

Notice that so far all negation is unary and all conjunction is binary. Now our sample sentence above was 'Every dog is a canine'. This was eventually paraphrased as 'Non: noncanine belongs to some dog', which would be symbolized as '$-(-C+D)$'. Let us distribute the external minus here into the parenthetical phrase: '$--C-+D$'. Application of Double Negation would yield: '$C-+D$'. What we have now is a formula with a minus and a plus flanked by two terms. The minus here is no longer unary and the plus is no longer binary (thus it can be suppressed to give us '$C-D$'). The distribution of an external unary minus into a phrase to yield an internal binary minus is analogous to such a distribution in arithmetic. Thus:

$$-(-3 + 2) = --3 - +2 = 3 - 2$$

How should we read '$C - D$'? Clearly: 'C belongs to every D'. In effect, then, we have introduced a new binary formative, 'belongs to every', which we symbolize by a binary minus, defined in terms of our basic unary minus and binary plus. So instead of paraphrasing 'Every dog is a canine' several times to get 'Non: noncanine belongs to some dog' (i.e. '$-(-C + D)$'), we can simply paraphrase it as 'Canine belongs to every dog' (i.e. '$C - D$').

Statements which are formed by connecting pairs of terms by the defined binary formative 'belongs to every' are amplified canonical sentences. An inspection of the four classical categorical statement forms shows that two are basic and two are amplified. *I* and *O* are basic; *A* and *E* are amplified. Thus:

I: Some *S* is *P*; *P* belongs to some *S*; $P + S$

O: Some *S* is not *P*; non*P* belongs to some *S*; $-P + S$

A: Every *S* is *P*; *P* belongs to every *S*; $P - S$

E: No *S* is *P*; Every *S* is non*P*; non*P* belongs to every *S*; $-P - S$

(Notice that by adopting the mathematician's convention of suppressing only unary pluses there is no danger of confusing the unary and binary readings of any sign. In the formulation of *E*, for example, the first minus is unary but the second can only be binary.)

We have introduced by definition the useful formative 'belongs to every'. But there are English formative expressions which are not basic but which are not given, as 'belongs to every' was, their own defined formative. An example is 'only... is/are...' A sentence like 'Only gods are immortal' is paraphrased as 'No nongods are immortal' and then as 'It is not the case that some nongods are immortal'. This is then paraphrased as 'Non: immortality belongs to some nongods', which is canonical and symbolized as '$-(-M + -G)$'.

Our symbolic language now consists not only of our basic unary minus and binary plus but a defined binary minus and unary plus. Any term, then, is either positive, preceded by a (possibly suppressed) plus, or negative, preceded by minus. Every phrase consists of a pair of terms connected by a positive (plus) or negative (minus) binary formative. Every phrase is either a compound term or a sentence. Just as our basic binary plus could form either compound terms from terms or sentences from terms (including other sentences), so can our defined binary minus. When plus is used to form a compound term from a pair of terms it is read as 'and', e.g. 'big and fat'. When it is used to form a sentence from a pair of terms it is read as 'belongs to some', e.g. 'Wisdom belongs to some man'. And when it is used to form a sentence from a pair of sentences it is read as 'and', e.g. 'Reason belongs to every man and wisdom belongs to every philosopher'. When minus is used to form a compound term from a pair of terms it is read as 'but not', e.g. 'big but not fat'. When it is used to form a sentence from a pair of terms it is read as 'belongs to every', e.g. 'Wisdom belongs to every philosopher'. And when it is used to form a sentence from a pair of sentences it is read as 'if', e.g. 'Wisdom belongs to every philosopher if wisdom belongs to every man'.

Today's standard systems of mathematical logic insist on a fundamental difference between the logic of sentences composed from nonsentential terms (the predicate calculus) and the logic of sentences formed from sentential terms, sentences (the propositional, sentential, calculus). By contrast our formal language recognizes the formal similarities between the two kinds of sentences. The formative 'belongs to some', which forms sentences from nonsentential terms, and the formative 'and', which forms sentences from sentences, can share a common symbolic expression because that expression, +, has just those formal features, e.g. symmetry, which the two formatives have. From a nonformal point of view it may matter whether the binary plus is read in one way or another (we can use our convention of uppercase and lowercase letters to help us here), but from a purely formal point of view there is no difference at all. Here are some examples of this parallelism.

B belongs to some A	q and p
$B + A$	$q + p$
B belongs to every A	q if p
$B - A$	$q - p$
non$B$ belongs to some A	non$q$ and p
$-B + A$	$-q + p$
non$B$ belongs to every A	non$q$ if p
$-B - A$	$-q - p$
Non: B belongs to some A	Non: q and p
$-(B + A)$	$-(q + p)$

Suppose we want to formulate a sentence which uses our binary formatives both as term connectives and sentence connectives, e.g. 'If every philosopher is wise then some logician is wise'. This is first paraphrased as the canonical sentence, 'Wisdom belongs to some logician if widsom belongs to every philosopher'. We might then symbolize this as: '$(W + L) - (W - P)$'. If any doubt remains as to whether the first binary minus here is to be read as a term or sentential connective we could adopt a convention for using different bracket types. For example, we could agree to use square brackets only around sentential phrases and angular brackets around compound terms. Thus '$[A + B] + [C + D]$' would symbolize 'Some D is C and some B is A' while '$\langle A + B\rangle + \langle C + D\rangle$' would symbolize 'Some D which is C is a B which is A'.

6. The introduction of defined formatives, along with some minor notational conventions have given our amplified formal language far greater expressive capacity than our basic language.

It is not yet the full logic of English, but a large number of English statements can be formulated by it.

We state now the rules of syntax for the amplified language.

(i) Any letter is a term

(ii) The result of prefixing a minus/plus to a term is a (negative/positive) term

(iii) The result of placing a minus/plus between two terms is a phrase

(iv) Any phrase is a term

We also have the following notational conventions.

(a) Any plus prefixed to a term may be suppressed

(b) Uppercase letters symbolize simple terms; lowercase letters symbolize sentential terms

(c) Square brackets mark sentential phrases; angular brackets mark compound terms

The derivation rules for the amplified language will include those rules already formulated for the basic language, Commutation, Association, Simplification and Double Negation. Moreover, new rules can be formulated on the basis of the formal properties of our new binary minus. Remember that the binary plus is symmetric and associative. The binary minus, however, is neither symmetric nor associative. But, unlike binary plus, it is transitive. Thus we can formulate a rule of inference which preserves the fact that from '$A - B$' and '$B - C$' we can derive '$A - C$'. Notice that such a rule would not be a rule of immediate inference, as our other rules are. It would be a rule which governs the inference of one sentence from a pair of sentences. It is a rule of mediate inference.

> Syllogism: Any term which belongs to all of another term belongs to that to which that other term belongs

In both of the sample inferences above a term ('A') which belongs to all of another term ('B'), thus the first premise, is concluded to belong to whatever that second term belongs (viz. every or some C), thus the second premise. The second term, the one cancelled out, is the scholastics' "middle" term. In arithmetic the following equation could be seen as applying Syllogism.

$$(2 - 1) + (1 + 3) = (2 + 3)$$

By adopting our notational conventions, then, we can model inferences by use of Syllogism simply by cancelling middle terms, i.e. pairs of terms within the scopes of opposite signs.

We add next an axiom which depends upon taking both of our negative binary formatives as reflexive.

> Axiom: Any statement of the form '$A - A$' is always true

Finally, we add one more rule of immediate inference, this one involving our binary minus. Recall that the binary minus was introduced by distributing an external unary minus into a phrase formed by connecting a pair of terms by a binary plus. Thus:

$$-(X + Y) = -X - Y \quad \text{and} \quad -(-X + Y) = X - Y$$

These kinds of equivalences call to mind the old rule of obversion. Consequently

> Obversion: Any phrase of the form '$-(X + Y)$' is equivalent to a phrase of the form '$-X - Y$'

We have seen that our amplified language excedes the basic language in expressive capacity. Fitted with our plus-minus algorithm it is easy to test many inferences for validity and to prove them. For example, consider the Cesare syllogism:

> No P is M
> Every S is M
> So, no S is P

We can symbolize it as

$$\frac{\begin{array}{c}-(M+P)\\ M-S\end{array}}{-(P+S)}$$

A necessary condition for the validity of such an inference will be that the algebraic sum of the premises must equal the conclusion, e.g.

$$-(M+P)+(M-S)=-(P+S)$$

Next consider the argument

Some *P* is *M*
Every *M* is *S*
So, every *P* is *S*

We symbolize this as

$$\frac{\begin{array}{l}M+P\\ S-M\end{array}}{S-P}$$

In this case, since the algebraic sum of the premises does not equal the conclusion—the argument is invalid. Finally, a proof of Cesare would look like this.

1.	$-(M+P)$	premise
2.	$M-S$	premise
3.	$-(P+M)$	1, Commutation
4.	$-P-M$	3, Obversion
5.	$-P-S$	2, 4, Syllogism
6.	$-(P+S)$	5, Obversion

7. If natural language has a logic (something assumed by all traditional logicians but denied by many modern mathematical logicians), then it ought to be possible to devise a formal language which models all kinds of statement-making sentences

and models inference patterns among them. In other words, it ought to be possible for the logician to construct a formal system which closely matches the expressive and inferential powers of a language such as English. Our amplified formal language is not such a system. In order to achieve such a system we need to find ways of formalizing statements of all kinds (including, for example, relational and identity statements). Our algorithm must also be able to model inferences involving all such kinds of statements. A system such as that would amount to a *full* formal language. We will suggest now ways to alter and augment the amplified language to produce a full language.

Let us begin by looking at the formalizing of statements. Consider a sentence of the general form '*p* and *q* or *r*'. The sentence is syntactically ambiguous, so that its truth or falsity will depend upon how it is formally analyzed—as a conjunction or a disjunction. Grammarians and logicians of course are quite skilled at disambiguating such sentences. But ordinary speakers are well-equipped to engage in a variety of formal tasks such as the disambiguation called for here. We use punctuation in writing and pauses and emphases in speaking in order to indicate the scopes of different formal expressions like 'and' and 'or'. One interesting way in which we might disambiguate our sentence is to "split" the second connective using one half to mark the start of its range. We make our sentence a disjunction by splitting 'or' into 'either... or'. Thus: 'Either *p* and *q* or *r*', a sentence clearly formalized in the normal fashion among today's logicians as '$(p \cdot q) \vee r$'. Had we intended our sentence to be a conjunction we could have placed the 'either' after 'and'. Thus: '*p* and either *q* or *r*', a sentence formulated as '$p \cdot (q \vee r)$'. This same kind of splitting holds for 'and' when it occurs as the second of a pair of sentential connectives in a syntactically ambiguous sentence. Suppose our original sentence looked like this: '*p* or *q* and *r*'. In this case our choice of making this a conjunction or a disjunction can be effected by splitting 'and' into 'both... and' and placing the 'both' at the beginning of the conjunction range. Thus 'Both p or q and r' would have the form '$(p \vee q) \cdot r$', while 'p or both q and r' would have the form '$p \vee (q \cdot r)$'.

For the purposes of disambiguating sentences like those above the choice of using either split or unsplit versions of sentential connectives like 'and' and 'or' is extremely useful. In cases where no formal ambiguity is threatened either choice will do. For example, there is only a stylistic difference between 'John will go to the party and Tom will go to bed' and 'Both John will go to the party and Tom will go to bed'.

The sentential connectives 'and' and 'or' are not the only ones which have split versions. Indeed, one sentential connective is usually used in its split form, viz. 'if... then'. Its two unsplit forms are 'if' and 'only if'. We could form an ambiguous sentence like 'p and q only if r', but the split 'if... then' is so much more normal for most speakers that we would probably only form one of its two possible interpretations: 'If p and q then r' (i.e. '$(p \cdot q)r$'), or 'p and if q then r' (i.e. '$p \cdot (qr)$'). We can say 'q if p' or 'p only if q', but we would usually say 'If p then q'. Another sentential connective has an unsplit form far rarer in ordinary discourse than 'only if'. Its unsplit version is 'neither... nor'. (Some natural languages, like Hebrew, do have an unsplit version.) Unsplit 'neither... nor' is the logicians' Sheffer stroke (call it 'nand').

Thus far we have seen that four quite ordinary sentential connectives can occur in both split and unsplit guises.

Table 1

Split	*Unsplit*
both... and	and
either... or	or
neither... nor	nand
if... then	only if/if

All of these expressions are used not only to connect sentences but to connect pairs of nonsentential terms as well. We say 'John is (both) a gentleman and a scholar', 'Tom is always (either) sleeping or resting', 'Neither dogs nor cats are allowed in the store', 'Some men are happy only if unmarried'. And the first three connectives are frequently used to connect pairs of terms, one or both of which are singular. Thus: '(Both) John and Tom were invited to the party', 'John took (either) Mary or Polly', 'Stars and the moon are visible at night', 'Polly loves neither Fred nor Tom'. So the expressions listed in Table 1 are used to connect all kinds of terms. We have as well in English another "connective" which operates on both sentential and nonsentential terms, viz. negation.

Let us concentrate on English binary connectives. Are there any others besides those listed in Table 1? Consider the sentence 'Some singers are performers'. Normally we do not think of the formal expressions here, 'some' and 'are' as connectives. But surely we could. The two terms 'singers' and 'performers' are more clearly connected by 'some... are' than 'a creature was stirring' is "connected" by 'not' in 'Not a creature was stirring'. Why not say that just as a pair of sentences can be connected by 'both... and' to form a syntactically more complex sentence, a pair of terms can be connected by 'some... are/is' to form a sentence? The mere fact that the material elements here are nonsentential terms cannot prohibit 'some... are/is' from being a connective. After all, we have already seen that in ordinary English usage all of our connectives can connect nonsentential terms as well as sentences. And, since we choose to treat 'some... are/is' as a term connective, then 'every... is' ('all... are') and 'no... is' must be taken as term connectives as well.

These additional connectives are clearly the split versions of our old unsplit connectives, 'belongs to some', etc. Our amplified language paraphrased all sentences with these split connectives as sentences with corresponding unsplit connectives. But in English the split versions are often more natural. For our full formal language we need to find a way to give a split symbolization for such connectives. And this in spite of the fact that both traditional and modern logicians have tended to favor the unsplit versions for logical analysis. One challenge, then, is to devise a system of split binary connectives which is as simple to use as the amplified system.

The logical syntax of phrases in the amplified language is quite simple. Pairs of terms, each of which is positive or negative, are connected by an intervening positive or negative binary connective to form a phrase. The logical syntax of phrases in the new full language is only slightly more complex. Again, pairs of terms, each of which is positive or negative, are connected by binary connective. But now the connective is split, one part preceding the first term, the other preceding the second term. Consider the classical *I* statement, e.g. 'Some *S* is *P*'. With the amplified system we first paraphrased this as '*P* belongs to some *S*', and then formulate it as '$P + S$', with an unsplit binary plus. Let us forego such paraphrase. We will split the old binary plus here into two pluses—one for 'some' and the other for 'is'. Moreover, this allows us to retain the natural word order of the English original. The two signs together constitute a term connective, forming phrases from terms. Considered separately, the first plus ('some') is a "quantifier"; the second plus ('is') is a "qualifier" ("copula"). So the logical syntax of the split connectives is simply the old subject-predicate syntax. Our subject is the quantified term, '$+S$' ('some *S*') and our predicate is the qualified term '$+P$' ('is *P*'). As with the amplified system, we have been suppressing most plus signs (but never those operating as quantifiers). Retrieving these for our *I* statement gives us the following.

$$+[+(+S) + (+P)]$$

Here the first plus is the unary positive sign on the entire sentential phrase; the second plus is the quantifier; the third plus is the unary sign on *S*; the fourth plus is the qualifier; and the fifth plus is the unary sign on P. The ordered pair consisting of the second and fourth (i.e. the quantifier and qualifier) constitutes the binary connective joining '$+S$' and '$+P$' to form the phrase.

The splitting of binary connectives promises more natural symbolic forms. We have seen this for I statements. Let us consider now the other classical categorical forms. An O statement, 'Some S is not P', can be easily formulated now as '$+S-P$', the abbreviation of '$+[+(+S) + (-P)]$'. A and E statements are the contradictories (negations) of O and I statements respectively. Initially, then, let us formulate them accordingly, i.e. A: '$-[+S - P]$' and E: '$-[+S + P]$'. Distributing external minuses now yields A: '$-S+P$' and E: '$-S-P$'. Thus, while the particular quantifier ('some') was symbolized by plus, the universal quantifier ('every') is symbolized by minus. Notice that the traditional form of E has always been 'No S is P', which itself abbreviates 'Not an S is P'. And this sentence has the form '$-[+S + P]$', with the external minus undistributed. By distributing it we have formed the equivalent obverse, viz. 'Every S is non$P$'. In summary, then:

I	Some S is P	$+[+(+S) + (+P)]$	$+S + P$
O	Some S is not P	$+[+(+S) + (-P)]$	$+S - P$
A	Every S is P	$-[+(+S) + (-P)]$	$-S + P$
E	No S is P	$-[+(+S) + (+P)]$	$-S - P$

Split connective notations such as '$+ \ldots +$' need not be interpreted only as nonsentential term connectives. They might just as well be construed as sentential connectives. For example, 'Both p and q', the split version of 'p and q' (i.e. '$q + p$') could by symbolized as '$+p + q$', where the first plus symbolizes the first part of the split connective ('both') and the second plus symbolizes the second part ('and'). As we will see, our ability to read connectives in this way, as applying to sentential and nonsentential terms alike, is due to the formal features shared by such pairs of formatives as 'some... is'/'both... and', 'every... is'/'if... then', and 'no... is'/'neither... nor'. In summary,

Table 2

Formula	*Term Reading*	*Sentential Reading*
$+X + Y$	Some X is Y	Both X and Y
$+X - Y$	Some X is not Y	Both X and not Y
$-X + Y$	Every X is Y	If X then Y
$-X - Y$	No X is Y	If X then not Y (Neither X nor Y)

Here are some sample sentences formulated using our split plus-minus notation.

Some boys are unwashed	$+B - W$
Every dog is loyal	$-D + L$
Not all philosophers are wise	$-[-P + W]$
Every man who is unwed is a fool	$-\langle M - W\rangle + F$
If some man is unwed, all girls are sad	$-[+M-W] + [-G+S]$
If it's raining then it's cold	$-r + c$
It's not both raining and not cold	$-[+r - c]$
It's not raining or it's cold	$-[-[-r]] - [-c]$
It's raining or it's cold	$-[-r] - [-c]$
It's neither raining nor cold	$-r - c$
If it rains then some cows are wet	$-r + [+C + W]$

Whatever effectiveness the amplified system might have had was due to the fact that the formal features of the formatives modeled those of their corresponding natural language expressions. Thus, the symmetry of 'and' and 'belongs to some' was preserved in the symmetry of the unsplit binary '+'. And just as neither 'only if' nor 'belongs to every' are symmetric, neither is the unsplit binary '−'. This preservation of formal features by the notation also characterizes our new formal language. The symmetry of '+ ... +' matches that of 'both... and' and 'some... is'. Likewise, 'neither... nor' and 'no... is' are symmetric and so is '− ... −'. Generally,

$$+X + Y = + Y + X \quad \text{and} \quad - X - Y = - Y - X$$

Expressions such as 'if... then' and 'every... is' are not symmetric, and neither is '$-$ ... $+$'. Later we will make use of such formal features in formulating derivation rules for the full formal language.

8. First, however, we must admit that while our new language is closer in syntax to English, we cannot yet say that its expressive power is much greater than that of the amplified language.

Sometimes in English we construct a nonsentential term using a binary connective read as a sentential connective. For example, we say 'Some men are both gentlemen and scholars', where the (split) connective, 'both... and', forms not a sentential phrase (sentence) but a compound term. Thus we would formulate our sentence as '$+M + \langle +G + S \rangle$'. In other words, expressions such as 'both... and', 'if... then', 'either... or' and 'neither... nor' can be used to form phrases which are sentences or phrases which are compound terms. Similarly, expressions which we have so far seen as forming phrases only from nonsentential terms, viz. 'some... is', 'every... is', and 'no... is', can also be used to form both sentential phrases and nonsentential phrases. In the latter case, however, such phrases are not compound terms—they are *relational* terms.

Unless we can formulate relationals using our new language we will be unable to match much of the expressive power of a natural language like English. Sentences such as 'Every boy loves a girl', 'Whoever draws a circle draws a figure' and 'Some people who write books are logicians' contain relational terms. Thus, at this stage, they seem to lie beyond the expressive powers of our system. We can formulate such sentences, however, once we recognize that relational terms are always constructed from a pair of terms (usually just simple ones) by means of an unsplit binary connective. Consider the sentence 'Some senator bribed some judges'. The subject term here is 'senator' and the predicate term is the relational expression 'bribed some judges'. Now 'bribed some judges' consists of a pair of terms connected by 'some'. This is clearly an unsplit binary connective. Using our old unsplit notation, we could symbolize the relational term by '$B+J$'. Likewise, we could symbolize, say, 'loves every girl' as '$L - G$'. Now in English we use these "quantifiers" in just this way, as unsplit binary connectives forming relational terms from ordered pairs of terms (ordinarily a transitive verb and a noun). Nevertheless, we want a uniform notational system, one which splits, now, all binary connectives. At the same time we want to retain a syntax close to that of English.

Let us take a relational term like 'bribed some judges' to have the general form: predicate-subject, where the qualifier is suppressed. It is easy to think of 'some judges' as a subject, i.e. a quantified term. And we can think of the predicate here as 'was bribing', with the qualifier, 'was', suppressed. The entire sentence is subject-predicate, where the predicate itself is a complex term (relational) of the form: predicate-subject. By splitting the binary connective forming the relational term (or, equivalently, by retrieving the suppressed qualifier) we can formalize all expressions involving relational terms by means of the split connective notation. Using { and } to indicate relationals, we could symbolize 'Some senator bribed some judges' as '$+S+\{+B+J\}$'. Here the '$+ \ldots +$' of '$\{+B+J\}$' is a split symbolization of the unsplit 'some' of 'bribed some judges'.

One problem is yet to be solved if we are to have an adequate system for formalizing relationals. We have seen that a sentence such as our sample above has the general form: subject-{predicate-subject}. Now the interpretation of the predicate is tied to the order of the two subjects here. 'Some senator bribed some judges' and 'Some judges bribed some senator' are quite different sentences. Let us call the subject of a predicate-subject relational term the "object". So our sentence has the form: subject-{predicate-object}. Sometimes when we reverse the order of subject and object in such cases the result is a different sentence all together; other times when we do so we form what grammarians call the "passive transformation" of the original sentence. Thus, while 'Some judges bribed some senator' is a different sentence all together, 'Some judges were bribed by some senator' is merely the logically equivalent passive transformation of 'Some senator bribed some judges'. The connection between the interpretation of a relational predicate and the order of subjects and objects is so important logically that we need to keep track of it in our notation.

From now on we will give a numerical subscript to each subject and each object. And we will subscribe to each relational predicate term the appropriate subject and object numerals in the order determined by our interpretation of that term. For example, we will symbolize 'Some senator bribed some judges' as '$+S_1 + \{+B_{12} + J_2\}$'. The order of subscribed numerals on the relational predicate term indicates that it is to be interpreted so that the first subscribed term bribed the second. 'Some judges bribed some senator' would then be symbolized as '$+J_2 + \{+B_{21} + S_1\}$' where the order of subscribed numerals on the relational predicate now indicates that what is symbolized by the term indexed by 2 bribed what is symbolized by the term indexed by 1. Finally, 'Some judges were bribed by some senator' is symbolized by '$+J_2 + \{+B_{12} + S_1\}$'. Note that here, as in the first case, the order of the numerical subscripts on the relational predicate term indicates that the senator did the bribing and the judges got the bribes. The passive transformation, then, is effected simply by altering the order of subjects and objects without altering the order of subscripts on the relational predicate term. Here are some further examples.

Every boy loves a girl	$-B_1 + \{+L_{12} + G_2\}$
Every boy loves every girl	$-B_1 + \{+L_{12} - G_2\}$
Whoever draws a circle draws a figure	$-\{+D + C\}_1 + \{+D_{12} + F_2\}$
Some people who write books are logicians	$+\langle +P_1 + \{+W_{12} + B_2\}\rangle + L$

A man gave a rose to a woman $+M_1+ \{+\{+G_{12} + R_2\}_{13} + W_3\}$

In this last case we could adopt a convention for amalgamating relational terms nested inside relational terms, fusing their subscribed numerals so as to preserve order. Thus we could simplify our formula as follows: $+M_1 + \{+G_{123} + R_2 + W_3\}$. We use this convention in the following examples.

A man gave a woman a rose	$+M_1+\{+G_{123}+W_3+R_2\}$
A woman was given a rose by a man	$+W_3+\{+G_{123}+R_2+M_1\}$
A rose was given to a woman by a man	$+R_2+\{+G_{123}+W_3+M_1\}$
A rose was given by a man to a woman	$+R_2+\{+G_{123}+M_1+W_3\}$

Notice that, because of the shifting order of subjects and objects, the 'G_{123}' is read as 'gave' in the first case, as 'was given... by' in the second, as 'was given to... by' in the third, and as 'was given by... to' in the fourth. All of the last five sample sentences are logically equivalent passive transformations of one another.

9. A full formal language such as the one we are after must be able to give a logical analysis of all kinds of natural language expressions. In particular, it must offer a means of formalizing categoricals, compound sentences (such as conditionals and disjunctions) and relational expressions. The old traditional syllogistic provided techniques for logically analyzing the first kinds of expressions only. Leibniz devoted much of his logical work to an attempt to incorporate into syllogistic all other kinds of expressions, thus making syllogistic a genuinely universal formal system. In our attempt to build such a system, what we have called a full formal language, we have found ways to formalize categoricals, compound sentences and relationals. Moreover, we have made use of just our unary and binary pluses and minuses. There is one other kind of natural language expression which cannot be ignored by an adequate logical system, viz. singular terms (and phrases involving them).

In the standard logic popular today the logic of singular terms is quite different from the logic of general terms.[1] It is not obvious to everyone however that the difference here reflects one which holds for singulars and generals in natural language. In a natural language like English general terms can appear in both subjects and predicates of sentences. Thus in 'Some singer is a performer' and 'Every performer is vain' the general term 'singer' occurs in the subject of the first sentence, the term 'vain' appears in the predicate of the second, and the term 'performer' occurs in the predicate of the first and in the subject of the second. It is this ability of general terms to occupy both subject and predicate positions which makes traditional syllogistic reckoning possible. General terms can be quantified (to form subjects); or they can be qualified (to form predicates). In our examples above 'performer' was qualified in the first sentence and quantified in the second.

General terms, as we have seen, can be qualified, quantified, negated, conjoined/disjoined (to form compound terms). These features were generally understood by traditional logicians and grammarians. None of these ideas however has survived as part of today's standard logic. According to the theory of logical syntax upon which modern mathematical logic is built, there are two kinds of sentences—those with no syntactical complexity at all ("atomic" sentences), and those with some syntactical complexity ("molecular" sentences). Atomic sentences always consist of a predicate and one or more subjects. Predicates are always general terms; subjects are always singular terms. Atomic sentences never contain any formative expressions. In particular, they never contain quantifiers, qualifiers, negators, conjoiners or disjoiners. The presence of any formative element is a guaranteed indication of syntactical complexity (molecularity). Such a logic gives no role at all to qualifiers. The other formatives, moreover, are never allowed to operate on nonsentential terms (even general ones). Quantification, negation,

[1] See Englebretsen, [6], [9] and [10].

conjunction, etc. can only apply to sentences to form syntactically more complex sentences. Thus 'John is a singer' is an atomic sentence consisting of two syntactically simple expressions: a predicate, 'is a singer' (symbolized 'S') and a subject, 'John' (symbolized 'j'). The sentence has the form 'Sj'. Sentences such as 'John is not a singer', 'John is a singer and a dancer' and 'Some singer is a dancer' are all molecular, constructed from atomic sentences and formatives on them. 'John is not a singer' is taken as the negation of 'John is a singer' and so is symbolized as '$\sim[Sj]$'. 'John is a singer and a dancer' is viewed as a conjunction of the two atomic sentences 'John is a singer' and 'John is a dancer' and is thus symbolized as '$[Sj] \cdot [Dj]$'. 'Some singer is a dancer' is seen as a quantifier applied to a molecular sentence. The quantifier is 'there exists some thing such that' and the molecular sentence is a conjunction of 'it is a singer' and 'it is a dancer'. It is formalized as '$(\exists x)(Sx \cdot Dx)$'. This is a clear example of how the modern logician's identification of predicates with general terms forces him or her into creating pronominal subjects. The traditional logician simply took 'Some singer is a dancer' to be a quantified term concatenated with a qualified term. The traditional logician saw this sentence as saying of some singer that it is a dancer. The contemporary logician sees it as saying of some thing that it is a singer and it is a dancer. One kind of logician sees it as about a singer, the other sees it as about a thing (a bare particular).[2]

Just as general terms can be quantified, qualified, negated and compounded, so too *mutatis mutandis*, can singulars. The distinctions between general, singular and sentential terms are semantic only. Syntactically they are all on all fours with one another. Thus the logic of singulars is not different from the logic of generals. And a logic which recognizes this enjoys the advantage of predicating (qualifying) singulars, thus elimi-

[2] Quine admits this explicitly in Quine [21], p. 165 and [22], p. 25.

nating the need to have a special "identity theory" appended to the main system.[3] Moreover, a logic such as ours, which recognizes sentences as terms, fit for all the logical roles reserved for any term, enjoys the not insubstantial advantage over today's standard logic of having a single common logic of terms and sentences. So a logic of sentences is not primitive and basic to a logic of terms. It is a part of the logic of terms. Such a logic gets by with one calculus instead of two.

So, singular terms can be quantified, predicated, negated and compounded. Since the first two of these notions have been well-surveyed[4] and since our thoughts concerning the notions of negated singulars have been fairly widely broadcast (see [6], [7], [8] and [10]), we merely note here that the negation of a singular term is not a singular. The mistaken belief that the negation of a singular term is singular (moreover, one which, if it denoted, could only denote an impossible object) has been the source of much confusion for many philosophers, logicians and grammarians. We note as well that singular terms in subject positions always have an implicit quantity, but, following a suggestion of Leibniz and Sommers, the quantity is arbitrarily either particular or universal. We will symbolize this "wild" quantity by '$*$', e.g. '$*S + W$' for 'Socrates is wise'.

It is wise to remember that not every use of 'and' is aimed toward conjunction. Consider the sentence 'Some logicians are scholars and gentlemen'. Here the compound term 'scholars and gentlemen' is the result of conjoining the two terms 'scholars' and 'gentlemen'. We could paraphrase the compound term as 'both scholars and gentlemen' by splitting the connective. Whenever a compound term of the form 'P and Q' can be paraphrased by thus splitting the connective we will call it a *genuine* conjunction. Not all conjunctions are genuine. Consider the sentence 'All the guests were men and women' (presumably,

[3] See [23] and chapter 6 of [26], and [5].

[4] Especially in the works by Sommers and Englebretsen.

no children were invited). Here the 'and' could not be split. The conjunction here is a *pseudo* conjunction. Pseudo conjunctions are, logically, disjunctions. We could paraphrase our sentence as 'All the guests were men or women'.

Let us turn now to singulars. Consider the quite unexceptional English sentence 'Al and Betty are together' and 'Al and Betty are happy'. Today's standard mathematical logician usually formalizes these as '*Wab*' and '*Ha* & *Hb*' (where '*W*' is read as 'is with' and '*H*' is read as 'is happy'). But now why is 'Al and Betty are together' taken as an atomic sentence with a relational predicate while 'Al and Betty are happy' is taken as a molecular sentence conjoining the two atomic sentences 'Al is happy' and 'Betty is happy'? Surface appearances suggest, contrarily, that 'Al and Betty are together' and 'Al and Betty are happy' should have the same logical structure. They certainly have the same grammatical structure (cf. 'Al and Betty are together and happy'). Our formal language can preserve this grammatical similarity. First, notice that each conjunction here is pseudo. In both sentences 'Al and Betty' is paraphrased in the same way and formalized as '$\langle -(-A) - (-B) \rangle$' ('what is either Al or Betty' or 'things which are either Al or Betty'). Now since 'Al and Betty' (read as a pseudo conjunction, i.e., a disjunction) is a subject term here, it is, implicitly quantified. In this case the logical quantity is universal (cf. 'Apples and oranges are fruits'). The two sentences have the forms: '$-\langle -(-A) - (-B) \rangle + T$' and '$-\langle -(A) - (B) \rangle + H$', where '$T$' is read 'together'.

Consider next a straightforward disjunction such as 'either Al or Betty' (as in 'The winner is either Al or Betty' and 'Either Al or Betty is a winner'). Here the compound term is simply transcribed as '$\langle -(-A) - (-B) \rangle$'. Suppose now that we have a sentence which we formulate as '$-\langle -(-A) - (-B) \rangle + T$'. How might that sentence look in English? Perhaps 'Both Al and Betty taught school'. And now we can begin to see the secret of singular compounds. Conjunctions of singulars, terms of the form 'A and B', are pseudo conjunctions. This is so because, assuming the uniqueness of each individual, a singular only applies to (denotes) one individual. A thing can be both short and fat, both red and square, or both even and prime, but nothing can be both Al and Betty, Paris and London, or 2 and the square root of 11. When it comes to compounds of singulars, such terms are either genuine disjunctions or pseudo conjunctions—none is a genuine conjunction. How then do we distinguish between these two kinds of disjunctions? As it turns out, when such terms are predicated (qualified) there is no difference—they are always simply disjunctions. Sentences formed as '$+T+\langle -(-A)-(-B)\rangle$' and '$-T+\langle -(-A)-(-B)\rangle$' would be read respectively as, e.g., 'Some teacher is either Al or Betty' and 'Every teacher is Al or Betty'. Conversion of each such sentence would result in a sentence of the general form '$+\langle -(-A) - (-B)\rangle + T$'. And it is the quantity of quantified compounds of singulars which determines whether they are genuine disjunctions or pseudo conjunctions. When the logical quantity is particular the compound is a genuine disjunction. When the logical quantity is universal the compound is a pseudo conjunction. Thus '$+\langle -(-A) - (-B)\rangle + T$' and '$-\langle -(-A) - (-B)\rangle + T$' could be read as 'Al or Betty teach' and 'Al and Betty teach' respectively.

But philosophers have had some confused and muddled things to say about compounds of singulars.[5] This has usually been due to their failure to appreciate two important points. First, they take all pseudo conjunctions of singulars to be genuine conjunctions. This has led them to say of subject terms such as 'Al and Betty' that they must name, denote, impossible objects, since what 'Al and Betty' denotes must have the properties of both Al and Betty; but many properties which Al and Betty have are logically exclusive (e.g., male/female, tall/short, bald/blond). The corrective here is simply the recognition that

[5] The confusion is found in [28] and [29]; [31]; [17]; [14]. Brief responses to [28], [29], [31] and [14] are found in [6], [7], [8] and [10].

no conjunction of singulars is genuine. Their second misunderstanding is more basic. The standard mathematical logician has taught them that all logical subjects must be singular. It follows from this that sentences like 'Al and Betty teach' and 'Al or Betty teach' either have singular subjects or are compounds of sentences with singular subjects. But, since treating compounds such as 'Al and Betty' and 'Al or Betty' as names of single individuals is nonsense, the functional expressions here must be seen as sentential connectives, contrary to surface appearances. The simple remedy in this case is to give up the dogma that all logical subjects must be singular.[6]

10. The idea that all logical formatives could be viewed as signs of opposition comparable to the plus and minus of mathematics provided us with the essential clue necessary for building a formal logical calculus for natural language. The basic plus-minus system led to the amplified system, and then to the full system. Our claim is that a system such as the one sketched here is simple, powerful and natural. This simplicity derives from the fact that all formatives are given a uniform symbolic representation as either plus or minus signs. Moreover, these signs, like those of arithmetic, are systematically ambiguous in that they have both binary and unary interpretations. We are able to model natural language statements by symbolic expressions which nearly perfectly match those statement forms when they are (or are paraphrased as) sentences using split binary connectives. This yields a formal language far more natural (i.e., closer in syntax to natural language) than that of the standard logical language taught in the schools today. Finally, by using the simple signs of arithmetic along with variable letters, we

[6] Note that while the traditional logician chose to admit only one logical subject for each sentence (but permitted logical subjects to have any number of referents), the contemporary logician chooses to admit only one referent for each logical subject, i.e., all logical subjects are singular (but permits sentences to have any number of logical subjects). A more detailed discussion of compound singulars is found in [13].

can use simple arithmetic reckoning to model the logical reckoning encountered in ordinary deduction. And we can do this for all sorts of such deductions—and all this power in a single, simple, uniform algorithm (see [26] and [11]).

The lexicon of our full system consists just of simple terms (singular or general). When used, each term is either positive or negative. Thus every used term is prefixed by either a plus or minus. Complex expressions (phrases) are always the result of a pair of used terms being connected by a binary connective. A phrase is considered a sentential term (sentence) whenever either the two constituent terms are themselves sentential or the split binary connective is construed as categorical-forming, i.e., when the first member of the connective is taken as a quantifier and the second is taken as a qualifier. Otherwise the phrase is a nonsentential, compound term. A phrase whose split binary connective is interpreted so that the first element of the connective is read as a qualifier and the second as a quantifier is a relational term. Since all used terms are either positive or negative, all used compound, sentential and relational terms are either positive or negative. Certain notational conventions are useful in the transcription of natural language expressions into formal language expressions. As in arithmetic, unary plus signs (except those interpreted as particular quantifiers) can be suppressed. Uppercase letters represent simple terms; lowercase letters represent sentential terms. Distinct sets of parentheses can be used to indicate compound, sentential and relational terms. Finally, in the case of relationals, we can use a system of numerical subscription to keep track of the interpretation of the relative predicate term and the order of subjects and objects. The general logical form of any statement will be as follows: a unary plus or minus indicating whether the sentence is positive or negative, the first element of a binary connective (read either as a particular quantifier or a 'both' if plus; as either a universal quantifier or an 'if' if minus), a unary plus or minus indicating whether the first term is positive or negative, a first term (simple, compound, relational or sentential), the second element of the binary connective (read as either a positive qualifier or an 'and' if plus; as either a negative qualifier or a then if minus), a unary plus or minus indicating whether the second term is positive or negative, a second term (simple, compound, relational or sentential).

The formulation of rules of inference for the full system, as for the basic and amplified systems, reflects the formal features of our connectives. Thus, since '$+ \dots +$' is symmetric, particular affirmative categoricals and their negations and conjunctions and their negations are all commutative. Since '$- \dots +$' is transitive, an appropriate universal affirmation or a conditional can be deduced from a pair of universal affirmations or a pair of conditionals respectively. These and other such formal features (viz., those mentioned above in our outline of the basic and amplified systems) allow us to use the following rules of immediate inference.

> *Commutation*: Any phrase of the form '$+X + Y$' is equivalent to a phrase of the form '$+Y + X$'
>
> *Association*: Any phrase of the form '$+(+X + Y) + Z$' is equivalent to a phrase of the form '$+X + (+Y + Z)$'
>
> *Simplification*: Any sentence of the form '$+p$' is derivable from a sentence of the form '$+p + q$'
>
> *Double Negation*: Any term of the form '$- - X$' is equivalent to a term of the form '$+X$'
>
> *Obversion*: Any phrase of the form '$-(+X + Y)$' is equivalent to a phrase of the form '$+(-X - Y)$'

Given the reflexivity of the connective '$- \dots +$', we add the following axiom.

> *Axiom*: Any positive statement whose split formative expression is such that its first element is negative, its sec-

ond element is positive and its two terms are identical is always true

Next, we add our rule for mediate inference.

Syllogism: From any pair of statements such that at least one has the form '$-X + / - Y$' derive a statement exactly like the other except that 'Y' has replaced 'X' at least once

The restrictions on mediate inference, syllogistic, validity are quite simple and are reflected in our rule. In effect, such an inference is valid just as long as (i) at least one premise is equivalent to a statement of the form '$-X + / - Y$', (ii) the second premise and conclusion have the same logical form, and (iii) the conclusion is the algebraic sum of the premises.

We conclude now with some examples of proofs using our new system.

Example 1. From 'No P is M' and 'Every S is M' derive 'No S is P'

1.	$-P - M$	premise
2.	$-S + M$	premise
3.	$-[+P + M]$	1, *Obversion*
4.	$-[+M + P]$	3, *Commutation*
5.	$-M - P$	4, *Obversion*
6.	$-S - P$	5, 2, *Syllogism*

Example 2. From 'Every circle is a figure' derive 'Every drawer of a circle is a drawer of a figure'

1.	$-C + F$	premise
2.	$-\{+D + C\} + \{+D + C\}$	*Axiom*
3.	$-\{+D + C\} + \{+D + F\}$	1, 2, *Syllogism*

Example 3. From 'Every boy loves some girl', 'Every girl adores some cat', 'All cats are mangey' and 'Whatever adores something mangey is a fool' derive 'What some boy loves is a fool'

1. $-B_1 + \{+L_{12} + G_2\}$ premise
2. $-G_2 + \{+A_{23} + C_3\}$ premise
3. $-C + M$ premise
4. $-\{+A_{23} + M_3\} + F$ premise
5. $-G_2 + \{+A_{23} + M_3\}$ 3, 2, *Syllogism*
6. $-G + F$ 4, 5, *Syllogism*
7. $-B_1 + \{+L_{12} + F_2\}$ 6, 1, *Syllogism*
8. $-\{+B_1 + L_{12}\} + F_2$ 7, *Association*

Example 4. From 'If p then q' and 'p' derive 'q'

1. $-p + q$ premise
2. $+p$ premise
3. $+q$ 1, 2, *Syllogism*

Example 5. From 'If p then q' and 'Not q' derive 'Not p'

1. $-p + q$ premise
2. $-q$ premise
3. $-[+p + [-q]]$ 1, *Obversion*
4. $-[+[-q] + p]$ 3, *Commutation*
5. $-[-q] - p$ 4, *Obversion*
6. $-p$ 5, 2, *Syllogism*

Example 6. From 'Tully is Cicero' derive 'Cicero is Tully'

1. $*T + C$ premise
2. $+C + T$ 2, *Commutation*

(interpreting the wild quantity in 1 as particular)

Example 7. From 'Tully is Cicero' and 'Cicero is Roman' derive 'Tully is Roman'

1. $*T + C$ premise
2. $*C + R$ premise
3. $*T + R$ 1, 2, *Syllogism*

(interpreting the wild quantity in 2 as universal)

11. Modern mathematical logic replaced traditional syllogistic logic about a hundred years ago. This change was rapid

and thorough. There were several reasons for the demise of the old syllogistic—some good, some bad. Not the least important of the good reasons was the fact that the new logic has powers of inference far beyond those of the old. The restricted scope of traditional syllogistic had been recognized clearly by logicians and philosophers for many centuries. Leibniz was one of the first, and most important, logicians to attempt to expand and modify the old syllogistic so that it could deal with the full range of inferences. He was particularly concerned with three kinds of inference which were beyond the scope of syllogistic: those involving singular terms, those involving compound sentences, and those involving relationals. What Leibniz recognized was that this restriction of power was due to syllogistic's lack of expressive power. What was required was a uniform, simple, perspicuous means of expressing all kinds of statements which enter into inferences. Traditional syllogistic was limited in its expressive capacity to simple categoricals.[7] By contrast, the new mathematical logic, initiated primarily by Frege, was able to give formal expression to a very wide range of expressions, to provide a relatively simple algorithm for manipulating formal expressions, and to build a system of reckoning all kinds of inferences involving such expressions.

The standard system of mathematical logic has no difficulty in giving a perspicuous analysis of inferences involving singulars, compound sentences or relationals. But, as we have seen, the system presented above, the full formal language, not only offers perspicuous analyses of such inferences, it does so in a simple, natural manner using a single uniform algorithm (the standard mathematical logic must make do with a propositional or sentential calculus, a predicate calculus and an identity theory). So our full system seems to match the standard system

[7] For a more detailed account of Leibniz's attempt at a full formal language see [4].

in inference power. In fact, our system *exceeds* the entrenched system in terms of inference power.[8]

As there were three kinds of inferences beyond the scope of traditional syllogistic, there are three kinds of inferences beyond the scope of today's standard mathematical logic.

> Case I: Plato taught Aristotle. So Aristotle was taught by Plato.

This inference is formalized by the standard system as

$$\begin{array}{r} Tpa \\ \therefore\ Tpa \end{array}$$

Where the ordinary speaker and the grammarian see the conclusion here as semantically equivalent but syntactically different from the premise, the modern logician is forced to give both statements the same logical form. The result is a trivialization of the inference involved. Our own system formulates the argument as

$$\therefore\ *\ A_2 + \{+T_{12} * P_1\}$$

This is a simple case of passive transformation, and is accomplished in our system by an application of *Commutation* (twice) and *Association* (once). The conclusion and premise may have the same truth conditions, but they are now clearly seen as formally distinct.

> Case II: Socrates taught a teacher of Aristotle. So, one whom Socrates taught taught Aristotle.

This argument is formalized by the standard system as

$$\begin{array}{r} (\exists x)(Tsx \cdot Txa) \\ \therefore\ (\exists x)(Tsx \cdot Txa) \end{array}$$

[8] This claim is made (along with examples such as those below) in [27].

Again, the standard system is powerless to exhibit the formal difference between the premise and conclusion. In our system the inference has the form

$$\therefore \; + \{*S_1 + T_{12}\} + \{+T_{23} * A_3\}$$

Grammarians take this to be a case of associative shift. While the modern logician sees it as a trivial reiteration, our logic recognizes the formal distinction between premise and conclusion here. The latter is derived from the former by an application of *Association*.

> Case III: Plato taught Aristotle with a dialogue. So Plato taught Aristotle.

The best that the standard system can do at formalizing this argument is this

$$\begin{array}{l} (\exists x)(Dx \cdot Tpax) \\ \therefore Tpa \end{array}$$

Here the two relational predicates are distinct—one is a three-place function, the other is two-place. For the inference to be valid there must be a hidden assumption of an analytic semantic tie between the two predicates. Our formalization retains a more natural syntax and the common sense view that 'taught' is univocal throughout its two uses in the argument.

$$\therefore \; * P_1 + \{+T_{123} * A_2\}$$

(It is important not to confuse the subscribed numerals here with the bound variable of the predicate calculus. Bound variables simultaneously keep track of reference and the order of subjects and objects with respect to the interpretation of the relational predicate. The numerals are needed only for the second of these tasks.) Our inference proceeds by the application of *Association* and *Simplification*.

12. The full formal language sketched out above seems to have certain important advantages over the standard mathematical logic. First of all, it is relatively simple. This simplicity is due in no small measure to its ability to give a uniform representation for all formatives in terms of the plus and minus signs of opposition used in mathematics. The resulting simplicity of expression, like that of mathematics, is dependent in part on the systematic ambiguity of such oppositional signs (as both unary and binary connectives). Secondly, the formal language is more natural than that of the standard system. This is because our syntax is built to be as close to that of natural language as possible. In contrast, the system of logical syntax upon which the standard logic is based was never motivated by a desire to preserve natural sentence forms. Indeed, the early authors of the system rarely refrained from denigrating the syntax of natural language, which was seen as hiding and obscuring a deeper logical syntax. Finally, our full logic surpasses the standard logic's ability to perspicuously analyze all kinds of inferences. Some quite simple inferences, accessible to our unschooled intuitions, are beyond its power.

Any theory which even seems to have important advantages over an established theory deserves at least a careful examination by relevant researchers. Our belief is that the full formal language of pluses and minuses does indeed have important advantages over today's standard mathematical logic. Our claim is that it at least seems to have such advantages.

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RESUMEN

Este artículo consiste en una introducción a los elementos de un lenguaje formal natural y claro (que por primera vez vislumbró F. Sommers). Mediante un sencillo algoritmo basado en la suma algebraica, este lenguaje se convierte en una importante herramienta con la fuerza necesaria para desafiar la lógica de predicados estándar.

[Traducción de Gabriela Montes de Oca V.]