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# ON NON-WELL-FOUNDED SETS

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Most of us have learnt to live with the fact that some singular terms do not denote. It would be fun were it otherwise, but no jolly fat man at the North Pole brings presents to good children at Yuletide; the name "Santa Claus" does not denote. Sets and classes may first have presented themselves to us as extensions of predicates, but it seems now the consensus that the paradoxes of set theory show that not all predicates have sets as extensions. Ramsey notwithstanding, many agree with the early Russell, and with Poincaré, in seeing an affinity between Russell's paradox and the paradox of the liar. So if the lesson of the paradoxes of set theory is that a predicate need no more have a set as extension than the name "Santa Claus" need denote someone, then perhaps the lesson of the liar paradox is that nothing answers to a liar sentence.[1]

To try this idea out systematically, we want some categories. Some singular terms denote, and "object" is a general term for the sort of thing singular terms succeed in denoting. We may say that there is no object denoted by the name "Santa Claus".

[1] Cf. W.D. Hart, "Russell and Ramsey," *Pacific Philosophical Quarterly*, 64 (1983), pp. 193–210.

Since "set" has become the favoured term for unproblematic extensions of predicates, we may say that the predicate "is not a member of itself" has no set as extension. Moore and Russell, from whom we inherit the term "proposition", thought of propositions as standing to sentences as objects stand to their names and as sets stand to predicates of which they are extensions. So the idea we are considering is that no proposition answers to, or, in another jargon, is expressed by, a liar sentence.

Making sense of this idea should include a good story of the structure of propositions and an account of why it encompasses none expressed by a liar sentence. That is a project Jon Barwise and John Etchemendy undertake in their book, *The Liar.*[2] One such story of propositions, the simplest Barwise and Etchemendy consider, they name for Russell. In fact Moore and Russell took propositions as primitive. But no matter, for the story Barwise and Etchemendy credit to Russell is one natural enough to demand consideration whether Russell actually told it or not.

If a singular term denotes, it denotes an object. If a predicate has an extension, its extension is a set. The simplest sort of sentence is a subject-predicate sentence got by applying a predicate to a singular term. A proposition is to be the extension of a sentence as objects and sets are extensions of singular terms and predicates. It would be tidy if the extensions of subject-predicate sentences were somehow composed out of the extensions of their subjects and predicates. The least demanding composition of them that keeps easy track of which is which is their ordered pair. In this way we arrive at the idea that the proposition expressed by a subject-predicate proposition is the ordered pair whose first member is the denotation of the subject and whose second member is the extension of

[2] Jon Barwise and John Etchemendy, *The Liar: An Essay on Truth and Circularity*, Oxford University Press, 1987.

the predicate. This story has a sweet little by-product, for we may say that such a proposition is true if and only if its first member is an element of its second. (But be warned that this story becomes less appealing as we allow for propositions of other forms, like negations and quantifications.)

How in this story should we characterize the liar? Here it might seem most natural to think first of a sentence like "This sentence is false." The subject of that sentence is a singular term that denotes that very sentence itself, and the sentence says of itself that it is false. But in our present story, truth and falsity are attributes not so much of sentences as of their extensions, the propositions they express. So an example of a liar like "This sentence is false" is ill-cast in our present story of propositions. These reflections suggest that we begin instead to characterize the liar from an example like "The proposition expressed by this sentence is false," or even "This proposition is false." In such examples the subject is a singular term purporting to denote a unique proposition, the proposition expressed by the sentence, and the sentence says of that proposition that it is false. In that way the examples can at least attribute falsity to, or deny truth of, things of the proper sort.

In any such example, a proposition will bear two relations to a sentence; it will be both denoted by the subject of the sentence and expressed by the sentence. But on the current story, the proposition expressed by a subject-predicate sentence is the ordered pair of the denotation of the subject and the extension of the predicate. So, since in the present examples the denotation of the subject is the proposition, expressed by the sentence, this proposition is an ordered pair that is its own first member. Indeed on the current story of propositions, we could define a self-referential subject-predicate proposition as an ordered pair that is its own first member (and whose second member is any set).

Wiener and Kuratowski independently taught us ways to construe ordered pairs in terms of ordinary sets.[3] For example, on Kuratowski's model, the ordered pair of $x$ and $y$ is the set whose members are the unit set of $x$ and the set whose members are $x$ and $y$. So a self-referential subject-predicate proposition is a set that is a member of a member of itself. Hence, to characterize the liar in the current story of propositions, we should cast propositions in a context that permits sets to be members of members of themselves.

Such a context is, in 1992, unorthodox. The dominant view of sets nowadays is the iterative conception. George Boolos has published a masterful exposition of this conception for philosophers.[4] To review it all too swiftly, we might start from the ordinal numbers. These are one way of generalizing the natural numbers (that is, the non-negative whole numbers, namely, 0, 1, 2 and so forth) into the infinite. There are two kinds of natural numbers. The natural number 0 has no natural number before it. Any other natural number is of the form $k + 1$, where $k$ is not only a natural number before $k + 1$, but also the natural number *immediately* before $k + 1$. Numbers having not only predecessors but immediate predecessors are called successors, and we write $k + 1$ for the successor of $k$. The least number principle, a basic law of natural numbers, says that if there is a number with a property, then there is a least number with that property. We hang on to this principle, but suppose that there is a number, called a transfinite ordinal number, greater than all natural numbers (now called finite ordinals). Then by the least number principle, there is a least ordinal greater than all natural numbers. Call it $w$. So $w$ has predecessors, but no immediate predecessor, for if $k$ were an immediate predecessor of $w$, $k$ would be natural and $w$ would be $k + 1$, so $w$ would be

[3] W.V. Quine, *Word and Object*, MIT Press, 1960, section 53, pp. 257–262.

[4] George Boolos, "The Iterative Concept of Set," reprinted in *Philosophy of Mathematics: Selected Readings*, Benacerraf and Putnam, eds., 2nd ed., Cambridge University Press, 1983, pp. 503–529.

natural. An ordinal like $w$ that has predecessors but no immediate predecessor is called a limit ordinal. (Every limit ordinal always has a successor, which has a successor, and so on.) We have to take steps to ensure a large supply of limit ordinals.

Because the ordinals satisfy a least number principle, they also satisfy inductive principles. So, for example, we can specify an operation on ordinals by giving its value at 0 outright, by giving its value at a sucessor $\alpha + 1$ in terms of its value at $\alpha$, and by giving its value at a limit $\lambda$ in terms of its values at the predecessors of $\lambda$. In this way we may define what are called the ranks. Let rank 0 be the empty set, the set that has no members. Let rank $\alpha + 1$ be the power set of rank $\alpha$, the set of all subsets of rank $\alpha$. For any limit $\lambda$, let rank $\lambda$ be the union of the ranks $\beta$ for $\beta$ less than $\lambda$, that is, the set of all things in rank $\beta$ for some $\beta$ less than $\lambda$. Then on the iterative conception, an object is a (pure) set if and only if it is a member of rank $\alpha$ for some ordinal $\alpha$. Since any set is a member of rank $\alpha$ for some $\alpha$, there is by the least number principle a least such ordinal $\alpha$ called *the* rank of the set. We can show that for any sets $x$ and $y$, if $x$ is a member of $y$, then the rank of $x$ is less than the rank of $y$; as it were, on the iterative conception, a set is always 'constructed later than' its members. It follows that if there were a sequence $x_n$ of sets such that for all $n$, $x_{n+1}$ is a member of $x_n$, then the sequence of ranks of these sets would be an infinitely descending sequence of ordinals, and, by the least number principle, there are no infinitely descending sequences of ordinals; so there are no such infinitely descending $\in$-sequences of sets. (If, in particular, there were a set $x$ that is a member of a $y$ of $x$, then the sequence with $x$ in odd places and $y$ in even places would be an infinitely descending $\in$-sequence, so on the iterative conception of sets no such $x$ and $y$ exist.) It follows from the fact that there are no infinitely descending $\in$-sequences that every non-empty set has a member disjoint from it, and conversely. For if there were such a sequence, the set of its terms would meet each of its members; and from a non-empty set meeting each of its members we could select an infinitely descending $\in$-chain.

The totality of sets in ranks is called the cumulative hierarchy; the tiers in the hierarchy are the ranks, and it is cumulative because when $\beta$ is greater than $\alpha$, rank $\alpha$ is a subset of rank $\beta$. The cumulative hierarchy is nowadays often taken to be the intended domain of set theory, and that domain with the membership relation restricted to that domain, to be the intended, or standard, model of set theory. The principle that every non-empty set has a member disjoint from it is (a version of) what is called the axiom of foundation (or, sometimes, regularity). We have seen that the cumulative hierarchy satisfies the axiom of foundation. The received formal statement of axiomatic set theory is called ZF, Z for Ernst Zermelo and F for Abraham Fraenkel, though by all rights Thoralf Skolem deserves a mention too. So the conventional wisdom about sets forbids telling the story above about propositions and the liar.

Actually, the argument we gave above to the effect that our version of the axiom of foundation (every non-empty set has a member disjoint from it) rules out infinitely descending $\in$-chains cannot be formalized in ZF, if ZF is consistent. For suppose ZF is consistent, say, because it comes out true in the cumulative hierarchy, where there are no infinitely descending $\in$-chain. Add to the language of ZF an infinity of new constants, one constant $a_n$ for each natural number $n$. Add to ZF each of the axioms of the form $a_{n+1} \in a_n$. Call this new theory $ZF^+$. If, $ZF^+$ were inconsistent, there would be a proof of a contradiction in it. But all proofs are finite, so only finitely many of the axioms of ZF and the new axioms of $ZF^+$ can occur in this proof in $ZF^+$ of a contradiction. Let $n$ be large enough that $a_1 \in a_0, \ldots, a_{n+1} \in a_n$ includes all the new axioms of $ZF^+$ in this proof. In the cumulative hierarchy let $x_0$ be the empty set and let $x_{k+1}$ be the unit set of $x_k$. Now let $a_0$, $a_1, \ldots, a_n, a_{n+1}$ denote $x_{n+1}, x_n, \ldots, x_1, x_0$ in the cumulative hierarchy. Then all the axioms in this proof come out true so interpreted in the cumulative hierarchy. So by the soundness of logic, the contradiction would have to be true in the hierarchy, which is impossible. Hence, by the completeness theorem, since $ZF^+$ is consistent, it has a model; moreover, this model may be taken to be an expansion of the cumulative hierarchy with an infinitely descending $\in$-chain. This sort of argument is due to Leon Henkin.

To tell the story above about propositions and the liar, we need another, different story of sets. At this juncture, Barwise and Etchemendy call upon Peter Aczel's gorgeous survey exposition of alternatives to the axiom of foundation.[5] To illustrate the idea, let us use dots —call them nodes— to represent (pure) sets, and let us use an arrow from a dot for set $x$ to a dot for set $y$ to represent $y$ being a member of $x$. (So arrows go back along membership.) In this way, for example, a dot or node with no arrows from it shows the empty set, while of two dots with just one arrow from the first to the second, the first shows the unit set of the empty set. In this way we can draw pictures of each of von Neumann's set theoretic reconstructions of the natural numbers, and of all the hereditary finite pure sets.

Now let us generalize. Take any old set. This set can come from anywhere, the cumulative hierarchy or even one of the realms of non-well-founded sets we are about to describe. Call the members of this set *nodes*. To make the set into what we will call a *graph* we add to it any old binary relation on it. For a binary relation on a set is a set of ordered pairs of members of it, and we will picture such an ordered pair as an *arrow* from its first member to its second. So a graph is a set of nodes and arrows between them. A *pointed* graph is a pair consisting of a graph and one of its nodes, which is called the point of the pointed graph. (So there are $n$ ways to make a graph with $n$

[5] Peter Aczel, *Non-well-founded Sets*, CSLI lecture notes no. 14, Center for Study of Language and Information, Stanford, 1988. The next eleven paragraphs draw heavily on Aczel's exposition.

nodes into a pointed graph.) A path in a graph is any sequence of its arrows that fit together head to tail; more abstractly put, it is a sequence $p_i$ of pairs in the relation on the nodes such that for each $i$, the second member of $p_i$ is the first member of $p_{i+1}$. An *accessible* pointed graph is a pointed graph in which for each node, there is a path from the point to the node (so such a path is a finite sequence of pairs of which the point is the first member of the first pair, and the given node is the second member of the last pair.)

When in a graph there is an arrow from a node $n$ to a node $m$, let us say that $m$ is a *child* of $n$. A *decoration* of a graph is a function that assigns sets to the nodes of the graph in such a way that the elements of a set assigned to a node are the sets assigned to the children of the node. Equivalently, if we write

$$n \longrightarrow m$$

to mean that $m$ is a child of $n$, then $d$ is a decoration of a graph $G$ if and only if for each node $n$ of $G$,

$$d(n) = \{d(m) | n \longrightarrow m\}$$

Note that decorations need not to be one-to-one; that is, they may assign the same set to several nodes. An accessible pointed graph is a *picture* of a set if and only if there is a decoration of the graph that assigns the set to the point of the graph.

Let us say that a path in a graph *stops* if and only if it is (a sequence of) finite (length) and, where $n$ is the second term of the last pair in the path, $n$ has no children. A graph is *well-founded* if and only if every path in it stops. This rules out both infinitely long paths and paths that loop back into themselves. Observe that every set in the cumulative hierarchy has a picture that is a well-founded accessible pointed graph. (For if $x$ is the set, let the nodes be the members of the transitive closure of $\{x\}$, and let an arrow go from a node $y$ to a node $z$ if and only if $z$ is a member of $y$. Any such picture is finitely deep, by the axiom of foundation; pictures of infinite sets are infinitely wide.) Conversely, any well-founded accessible pointed graph is a picture of a (unique) set in the cumulative hierarchy. (For to decorate such a graph we must assign to childless nodes the empty set, and to a node with children we must assign the set of sets assigned to its children.)

But well-founded graphs are pretty special, and maybe it is narrow-minded to think that only these among accessible pointed graphs picture sets. Perhaps every accessible pointed graph pictures a set. Suppose so. Then, for example, the graph consisting of a single node from which an arrow starts and ends pictures a set that is its own unit set. Observe too that though self-membership and non-self-membership occur, logic guarantees that no graph has a node whose children are all and only the nodes that are not children of themselves. More to the present purpose, recall that the proposition that Socrates is bald was to be the ordered pair of Socrates and the extension of the predicate "is bald". Letting "$F$" substitute for any predicate with an extension, a proposition that says of itself that it is $F$ was to be an ordered pair that is its own first member and whose second member is the set of $F$s. Let $H$ be an accessible pointed graph picturing the set of $F$s. Let $G$ be an accessible pointed graph from whose point $p$ descend two arrows, one to each of two nodes $n$ and $m$; let an arrow go from $n$ to $p$, and from $m$ let one arrow go to $p$ and another to the point of $H$. Then $G$ is a picture of a proposition that says of itself that it is $F$. Simple, isn't it?

But there is a glitch. The axiom of extensionality says that sets with the same members are identical. This axiom is as received as axioms of set theory get. The glitch is not that non-well-founded sets force us to deny extensionality, which would put us beyond the pale. To remain within, we must decorate nodes with the same children with the same set, and this we insist upon. But in the cumulative hierarchy of pure sets, extensionality suffices to fix the identities of sets. For it fixes the uniqueness of the empty set outright, and progressing up the hierarchy extensionality fixes the identities of sets of higher rank inductively by the identities of their members. In an applied hierarchy that began not from the empty set but from some set of individuals or urelements, we might have other principles for the identities of those individuals, and then those principles with extensionality would fix the identities of all the sets in the applied hierarchy inductively too. But where a set is its own unit set, there is no fixing the identity of the set's member before (at a lower rank than) that of the set itself. So inductive structure, like extensionality, from members of sets to sets seems insufficient to fix the identities of non-well-founded sets. In this way, non-well-founded sets are a significant departure from the conventional wisdom about sets.

Aczel has the grace and good taste to devote the lion's share of his monograph to the issue of the identities of non-well-founded sets. (Barwise and Etchemendy do not acknowledge the issue in their book.) To focus the issue, let $ZF^-$ be a formalization of the usual axioms of Zermelo-Fraenkel set theory (with choice but) without the axiom of foundation. To $ZF^-$ we will consider adding various principles all of which agree that every accessible pointed graph pictures a set but which disagree with each other about how the identities of non-well-founded sets pictured by non-well-founded such graphs come out. For a considerable range of such principles, Aczel shows that if ZF is consistent, then so is the corresponding extension of $ZF^-$. The basic idea of such relative consistency proofs is straight-forward. We make graphs from nodes that are sets in the cumulative hierarchy, and then we replace (or, in the jargon, identify) perhaps non-well-founded sets with equivalence classes of their graphs, where the underlying equivalence relation encodes one of the several principles about when non-well-founded sets are identical, that is, when different non-well-founded graphs picture the same set. (Actually, the equivalence classes are too big to be sets, so we cut each down to the *set* of such graphs of lowest rank.) Each of these worlds includes (isomorphs of) all the well-founded-sets of the cumulative hierarchy plus a treasury on non-well-founded sets, different treasuries reflecting different ways of counting such sets. Aczel shows us that we have an embarrasing freedom in how we can count such sets.

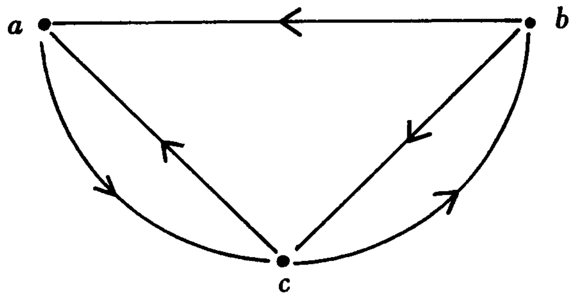
Suppose, at one extreme, that we want to maximize differences (minimize identities) between sets. There is a limit here. A graph is *extensional* if and only if nodes with the same children are identical. A decoration of a graph is *exact* if and only if it is one-one, that is, it assigns different sets to different nodes. By the axiom of extensionality, only extensional graphs have exact decorations. (For if a graph is not extensional, two different nodes in it have the same children. An exact decoration would assign these nodes different sets. But each must be decorated with the set of decorations of its children, so since they have the same children, they must be decorated with the same set.) Exact decorations draw distinctions, so the most we could do to maximize differences would be to insist that each extensional graph has an exact decoration. Aczel credits this principle to Boffa in the 1970s. (Actually, Boffa's axiom is somewhat stronger, and much more complex, than this principle.) To illustrate, take any set of nodes, and from each node draw a single arrow circling back to that node. This graph is extensional, so by the principle it has an exact decoration. In effect, we get as many non-well-founded sets that are their own unit sets as there are sets in the cumulative hierarchy, which is a lot.

Now suppose we want, at another extreme, to maximize identities (minimize differences) between sets. To do so, we might suppose that every graph has a *unique* decoration. (This is the alternative to the axiom of foundation with which Aczel leads off, and the only one Barwise and Etchemendy state. It seems to have been stated first by Forti and Honsell in the early 1980s.) A graph with a single node and a single arrow from that node looping back to it still guarantees the existence of a set that is its own singleton. But if there were two such sets, they would provide two different ways to decorate this graph, so now there is exactly one set that is its own singleton. Aczel calls this set $\Omega$. An accessible pointed graph is a picture of $\Omega$ if and only if every node in the graph has a child. For $\Omega$ suffices to decorate such a graph, and in any picture of $\Omega$ every node must have a child. So on this alternative to foundation, the infinitely descending accessible pointed graph consisting of nodes $n_0, n_1, \ldots$ and arrows just from $n_k$ to $n_{k+1}$ is also a picture of $\Omega$, albeit inexact, while on Boffa's alternative, it takes infinitely many sets to decorate this graph.

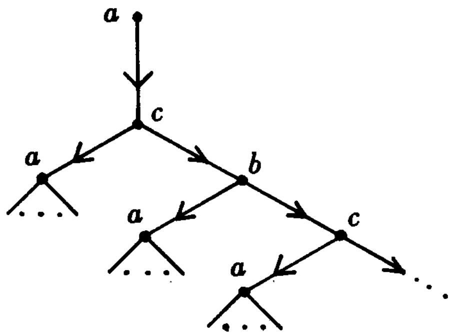
There are intermediates between these two extremes. It is easy to convert any decoration of a graph into a decoration of another graph isomorphic to the first. Perhaps one might want to say that if accessible pointed graphs are isomorphic, then the sets they picture are identical. But we must resist the converse. Let $G$ be a graph with one node and one arrow from it looping back to it. Let $H$ be a graph with two nodes, one arrow from the first to the second, and another from the second looping back to it. $G$ and $H$ are not isomorphic. If nodes of a graph have the same children, they must be decorated with the same set, so since the two nodes of $H$ have the same children, they must get the same decoration. The lower node of $H$ has to be decorated with a set that is its own singleton, which is also the sort of set that has to be used to decorate $G$. So $G$ and $H$ picture the same sets, even though they are not isomorphic. At this point Aczel modifies an idea of Paul Finsler's from the 1920s to get a necessary and sufficient condition for accessible pointed graphs to picture the same set. Let $G$ be an accessible pointed graph with point $a$. Let $G^*$ be the accessible pointed graph with the nodes and arrows of $G$ on paths starting from children of $a$, together with a new node and an arrow from this node to each child of $a$. (Note that if $a$ does not lie on a path starting from one of its children, then $G$ is isomorphic to $G^*$, but if $a$ does lie on such a path, then $G^*$ is $G$ plus the new nodes and arrows.) Aczel's modification of Finsler's idea is that $G$ and $H$ picture the same set if and only if $G^*$ is isomorphic to $H^*$. Let us postpone showing how this differs from Forti's and Honsell's idea.

Yet another idea comes from Dana Scott in 1960. A *tree* is an accessible pointed graph in which for each node there is a *unique* path from the point to the node. We sketched above a way to draw for each set a picture of it; Aczel calls this the cannonical picture of the set. The cannonical picture of a set is not in general a tree; it will fail to be a tree if, for example, two members of the set meet. But every picture of a set can be *unfolded*, as Aczel puts it, into a tree picture of the same set. Let $G$ be an accessible pointed graph with a point $a$. Form the tree whose nodes are all the paths through $G$ starting from $a$, and whose arrows go from, and only from, a path of the form $a \longrightarrow \cdots \longrightarrow a_n$ to a path of the form $a \longrightarrow \cdots \longrightarrow a_n \longrightarrow a_{n+1}$ (so arrows go from paths to their immediate extensions). The point, or root, of this tree is the path $a$ of length 1. (Counting $a$ alone as a path in $G$ is an extension of our earlier use of "path".) Any decoration $d$ of $G$ induces a decoration $D$ of the tree, for if the path $a \longrightarrow \cdots \longrightarrow b$ is a node of the tree, let $D(a \longrightarrow \cdots \longrightarrow b)$ be $d(b)$. So the unfolding of an accessible pointed graph pictures any set pictured by the graph. Aczel calls the unfolding of the cannonical tree picture of a set the cannonical tree picture of the set; the cannonical picture is always exact, but the cannonical picture is usually not. To illustrate, the unfolding of a graph with one node and one arrow from it looping back to it is isomorphic to the graph with nodes $n_0, n_1, \ldots$ and arrows from $n_k$ to $n_{k+1}$. For any accessible pointed graph $G$, let $G^t$ be the unfolding of $G$ into a tree. Scott's idea can be put by saying that accessible pointed graphs $G$ and $H$ picture the same set if and only if $G^t$ is isomorphic to $H^t$.

The next order of business is to see that these ideas differ extensionally. To show this, let $G$ be




Let $Ga$ be $G$ with point $a$, let $Gb$ be $G$ with point $b$, and let $Gc$ be $G$ with point $c$. On Forti's and Honsell's idea, the unique decoration of $G$ assigns $\Omega$ to every node, since each node has a child. Next, the unfolding of $Ga$ looks like




The unfoldings of $Gb$ and $Gc$ are isomorphic subtrees of $(Ga)^t$. So on Scott's idea, $G$ pictures a set $x$ (for $b$ and $c$ both) and a set $y$ (for $a$) such that $x \neq y$, $x = \{x, y\}$ and $y = \{x\}$. Next it is not difficult to check that no two of $(Ga)^*$, $(Gb)^*$ and $(Gc)^*$ are isomorphic, so on Finsler's idea, $G$ pictures three pairwise different sets $x, y$ and $z$ such that $x = \{z\}, y = \{x, z\}$ and $z = \{x, y\}$. Finally, consider two nodes each with a single arrow back to it. Boffa says this is a picture of two sets, but Forti, Honsell, Finsler and Scott all say it is an inexact picture of one set. So any two of our four alternatives to foundation are inconsistent with each other. In general, Boffa makes the most distinctions, Finsler, the next most, Scott, fewer still, and Forti and Honsell, fewest.

Aczel shows us a wealth of criteria of identity for non-well-founded sets. Indeed, that wealth may be an embarrasment of riches, perhaps counterfeit. For, to cite a precept Quine mentions, no entity without identity.[6] So if we have no basis for choice among competing criteria of identity for non-well-founded sets, then, perhaps, we do not know what such sets are, or would be if there were any. Perhaps we do not understand what Aczel (and Barwise and Etchemendy) are talking about well enough to decide sensibly whether to go along or stay home.

The jargon "criteria of identity" should be used circumspectly. To be sure, the number of $F$s is the number of $G$s if and only if there is a one-to-one correspondence between the $F$s and the $G$s, and well-founded sets are identical if and only if they have the same members. But this contrast does not show that identity is a patchwork, that it is one thing for numbers but another for well-founded sets, as perhaps pumps are one thing in shifting fluids but another on women's feet. Everything is what it is and not another thing, as Bishop Butler put it. Identity is identity, the smallest reflexive relation, as Warren Ingber[7] once put it. It is just that for some kinds of things, like numbers of $F$s and well-founded sets, identity is co-extensive with (another, in intension) relation between things of that kind, and mastery of such an equivalence is central to grasp of the kind.

To illustrate, in the good old days we were taught in school that two points determine a straight line and that three non-collinear points determine a plane. These facts can be put as

[6] W.V. Quine, "Speaking of Objects," reprinted in *Ontological Relativity and Other Essays*, Columbia University Press, 1969, p. 23.

[7] Once a graduate student in philosophy at the University of Michigan, and now a lawyer.

criteria of identity. Straight lines are identical if and only if there are two different points that lie on the lines. Flat planes are identical if and only if there are three different points on the planes such that no line passes through all three points. We can generalize these criteria on both dimensional sides. Points are identical if and only if there is one point lying on the points, and $n + 1$ dimensional hyperplanes are identical if and only if there are $n + 2$ different points on the planes and no two combinations of $n + 1$ of these points determine the same $n$ dimensional hyperplane. There is, of course, a legitimate question for each of these criteria whether it (and especially the last, inductive, generalization) is true, and it would utterly misconstrue these questions to take these criteria as an infinity of separate clauses in an unsurveyable 'definition' of identity. Besides, the value, such as it is, of these criteria is what they reveal via a prior grasp of identity about straight lines, flat planes and the cross-dimensional affinity between straightness and flatness.

Quine's mention of the precept "No entity without identity" differs from our use of it. He meant that because we have *no* clear criteria of identity for properties or propositions, we do not know what talk about such things is about. Our worry was that because we have, in contrast, *four* mutually inconsistent criteria of identity for non-well-founded sets, and no evident basis for picking a winner from them, we do not know what talk about such sets is about; criteria of identity for a single kind should be extensionally equivalent for there to be an extension, a kind.

But maybe we can come up with the right choice. Two thoughts might strike one here. First, among the more familiar sets of the cumulative hierarchy, it is the axiom of foundation that says that for only the well-founded graphs do there exist decorations, while it is the axiom of extensionality that says these decorations are unique. So if we are to drop foundation but keep extensionality, then since in the well-founded case, extensionality is uniqueness of decoration, perhaps uniqueness of decoration is the right way to lift extensionality into non-well-founded sets. Such considerations would favour Forti's and Honsell's axiom, and might account for Barwise's and Etchemendy's silent selection. Conservative systematicity makes sense, for surely it would be wasteful and less organized to flee what has served us well needlessly. But conservatism will not be the whole story if we are to abandon the foundation of our fore-fathers, and it seems sensible to direct our departure by our motives for leaving. We wanted to represent self-referential propositions set theoretically, and it might feel like manifest destiny to fathom a unique criterion of identity for non-well-founded sets in just that motive for departure. Thus the second thought that might here strike one is the question what, if anything, self-referential propositions have to say about the identity condition of non-well-founded sets.

So we might ask intuition whether self-referential subject-predicate propositions with a common predicate, like

> This proposition is expressed in seven words

and

> This proposition is expressed in seven words,

are the same or different. But intuition dithers. Perhaps it sketches two copies of a pointed graph with four nodes; in each there is an arrow from the point, node 1, to nodes 2 and 3, an arrow from each of nodes 2 and 3 to 1, and from 3 an arrow to the point of a picture of the set of all propositions expressed in seven words. Perhaps Finsler's and Scott's ideas seem too unmotivated to intuition, and it discards them. But the extremes remain. Will intuition say with Forti and Honsell that each graph has a unique decoration so they picture the same proposition, or will it say with Boffa that their joint graph overlapping just in its subpicture of the set of propositions expressed in seven words has an exact decoration, so they picture different propositions? Intuition leaves us in metaphysical suspense.

Confronted with this irreality, one might reflect that our only access to propositions is through sentences expressing them and that demonstration of sentence types is deferred[8] from their tokens, so perhaps we would do better to compare

> The proposition expressed by the sentence demonstrated by this token is expressed in fifteen words

with

> The proposition expressed by the sentence demonstrated by this token is expressed in fifteen words,

or to compare

(1) The proposition expressed by sentence (1) is expressed in eleven words

with

(2) The proposition expressed by sentence (2) is expressed in eleven words.

But different tokens of the same demonstrative sentence type might or might not express the same proposition, and what is to settle whether (1) is (2) or not? We are left hanging. Could any of these pairs differ from each other in truth value? Yes, perhaps, if it is a pair of two different propositions, but no, if not. Intuition is dumb.

It might seem frustrating to close so aporetically, but a recognition of loss is a gain. Perhaps, as Quine has long urged, it is in the nature of propositions to elude a lucid grasp. So far, at any rate, propositions and non-well-founded sets seem well matched in obscurity.[9]

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[8] W.V. Quine, "Ontological Relativity," reprinted in *Ontological Relativity and Other Essays*, Columbia University Press, 1969, pp. 40 ff.

[9] My opportunity to think about non-well-founded sets arose from Raúl Orayen's generous invitation to lecture on the subject at the Instituto de Investigaciones Filosóficas at UNAM in September 1991. I am indebted to Professor Orayen and his colleagues and students for a wonderful and stimulating visit to Mexico.

RESUMEN

En *The Liar*, Barwise y Etchemendy proponen una construcción de las proposiciones de acuerdo con la cual la proposición expresada por una oración de la forma sujeto-predicado es el par ordenado cuyo primer miembro es la denotación del sujeto y cuyo segundo miembro es la extensión del predicado. Aplicado a oraciones autorreferenciales como "la proposición expresada por esta oración es falsa", esto conduce a un par ordenado que es a la vez el primer miembro de dicho par. Dada la explicación habitual de los pares ordenados, esto conduce a un conjunto que es miembro de sí mismo. Tales conjuntos no son aceptados por las teorías corrientes, inspiradas en el enfoque iterativo de los conjuntos, que el autor describe sintéticamente. Pero hay un enfoque alternativo que sí acepta la existencia de conjuntos como el antes mencionado. El autor sintetiza las ideas fundamentales de esta teoría alternativa, expuesta en *Non-well Founded Sets*, de Aczel. Se muestra luego que la mera extensionalidad no basta para fijar la identidad de los conjuntos no-bien-fundados y que además existe una amplia variedad de alternativas para completar un criterio de identidad. Se argumenta que no hay razones firmes para preferir una de estas alternativas. Una comparación con las proposiciones muestra algo similar y el autor finaliza sugiriendo que las proposiciones y los conjuntos no-bien-fundados adolecen de oscuridades paralelas.

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