ON AN ALLEGED PARADOX OF CONSISTENCY AND MATERIAL IMPLICATION*

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In a paper which is both penetrating and suggestive in many ways,[1] Mr. Theodore C. Denise has called attention to a new puzzle in the theory of material implication.

> Given a sequence of valid arguments from ordinary discourse of the form $p \therefore q$, and $q, r \therefore s$, the p and q statements consistent and the q, r and s statements consistent, does it follow that the argument of form $p, r \therefore s$ is valid and that the p, r and s statements are consistent? If innocent of the logic of material implication, we would not, I think, hesitate to say it does; if guilty, we say only that the argument is valid and that its component statements may or may not form a consistent set . . . For some, a logic so "paradoxical" can be employed only with misgivings. (p. 62).

But just how paradoxical is this?[2] Obviously the paradox, if any, is not the kind of thing we associate with Russell or

* I am indebted to Professors Wilfrid Sellars, Charles Landesman, and Colin B. Wright for valuable comments on earlier versions of this paper.

[1] "Material Implication Re-examined", *Mind* xxi, No. 281, January 1962, pp. 62-68.

[2] If one is looking for a paradox of material implication as regards consistency, it is, of course, not hard to find one. It is well known that material implication is a *connected* relationship; for $(p \supset q) \vee (q \supset p)$ is a tautology. If we substitute $\sim p$ for q in this expression, it follows that $(p \supset \sim p) \vee (\sim p \supset p)$. Anyone who is innocent of 'the logic of material implication' would surely be reluctant to say that a proposition can 'imply' a statement which is contradictory to it, unless perhaps he is a Hegelian; and I think by almost any reasonable criterion a statement will be considered inconsistent with its own contradictory. Yet our formula apparently tells us that given any statement and its contradictory, one of them will materially imply the other. We even know which of them will be the *implicans*—namely, the one which is false. But we also know that no logician who is really a gentleman and well informed on these matters is likely to take unscrupulous advantage of this. And of course it would be quite out of line with the spirit of Denise's article for him to do so.

Burali-Forti; it lies rather in the simple discovery that we might be wrong in what we might intuitively expect. It does not lie in the fact that the validity of the third argument-form is required by that of the other two; for this remains unchallenged. But Denise shows that instances of these argument-forms can be constructed in such a manner that though each of the first two arguments is composed of statements which are consistent with one another, the third argument is composed of statements which are *in*consistent with one another in spite of this threefold validity. He suggests that this should be disturbing to anyone who is 'innocent of the logic of material implication'.

Now Denise has carefully avoided any explicit use of material implication in stating his problem. If, however, we are to show the relevance of his remarks, we can hardly be so squeamish. One rather obvious procedure is to replace each of his three argument-forms by the implication statement-forms corresponding to them. We then find that the following four conditions are to be taken as given:

(1) $p \supset q$;
(2) p and q are consistent with one another;
(3) $(q \cdot r) \supset s$;
(4) q, r, and s are consistent with one another.

Our problem then is to discover whether it follows that

(5) $(p \cdot r) \supset s$; and
(6) p, r, and s are consistent with one another.

By constructing a standard truth-table, or using the principles of exportation and the transitivity of '$\supset$', we can easily show that condition (5) is derivable from conditions (1) and (3) taken together, and is materially implied by them. But how about the other conditions?

Denise fails to state any criterion for consistency. Nor does he even suggest what sort of criterion would be appropriate to what he calls 'the logic of material implication'. While it is easy to define 'consistency' within a system of modal logic such as that of Lewis or von Wright, the concept of consistency seems to be strictly extrasystematic or metalinguistic if we are dealing with the kind of system Denise presumably has in mind. Metalinguistic criteria for axiom systems, such as those which Hilbert and Post have proposed, seem hardly appropriate here. But it is probably not inappropriate to suggest that a set of statements is 'consistent' if and only if the column for their conjunction in a standard truth-table contains at least one 'T', or, to put it more simply, if and only if there is at least one row in the truth-table which contains a 'T' for each statement in the set.[3] In the construction of such a truth-table, however, it is not enough to assign one column of 'T's and 'F's to each statement in the set, for instance to Denise's *p, q, r,* and *s.* If this were done, *any* set of statements could be proved consistent. The criterion becomes differentially effective only if at least some of these statements themselves are complex, and a column is assignated to each of the component statements into which they can be analyzed.

For if *p, q, r,* and *s* remain unanalyzed and a column of 'T's and 'F's is assigned to each of them according to the usual procedure, their truth-table will always have one row in which all of these are given the value 'T', and this in itself will guarantee that there will be at least one 'T' in the column for any conjunction of these taken severally, so that conditions (2), (4), and (6) will all be automatically met. And if these conditions all hold, it is clear that conditions (2) and (4) materially imply condition (6). Since (5) is materially implied by (1) and (3), and since (6) is materi-

[3] See, for instance, Patrick Suppes, *Introduction to Logic,* 1957, p. 38. Notice that it would be a mistake to define 'consistency' by the easy procedure of denying that *p* and *q* are 'incompatible' in Sheffer's sense, though this could be done within the system; for this would amount to asserting their conjunction. Nor would it be any better to require that they be materially equivalent. Both these criteria are much too strong to be applicable here.

ally implied by (2) and (4), it seems obvious that 'the logic of material implication' requires that condition (1) to (4) must materially imply conditions (5) and (6) just as those who are 'innocent' of that logic would supposedly expect.

But Denise does not allow p, q, r, and s to remain unanalyzed. He seeks to establish his paradox in two ways. His first procedure is to substitute P for p, $(\sim P \supset Q)$ for q, $\sim P$ for r, and Q for s in the three argument-forms, winding up correctly with the following set of arguments:

(7) $P \therefore \sim P \supset Q$; $\sim P \supset Q, \sim P \therefore Q$; and $P, \sim P \therefore Q$.

He then starts over again with the original argument-forms, and substitutes P for p, $(\sim Q \supset P)$ for q, $\sim P$ for r, and Q for s. He thus obtains the new set of arguments:

(8) $P \therefore \sim Q \supset P$; $\sim Q \supset P, \sim P \therefore Q$; and $P, \sim P \therefore Q$.[4]

If one replaces the three arguments in (7) by the corresponding implicational expressions, one obtains the following truth-table:

p P	q $\sim P \supset Q$	r P	s Q
T	T	F	T
F	T	T	T
T	T	F	F
F	F	T	F

(1)

$p \quad \therefore \quad q$ $P \supset (\sim P \supset Q)$
T
T
T
T

$q \quad , \quad r$ $(\sim P \supset Q) \cdot \sim P$
F
T
F
F

[4] Denise, *loc. cit.*

(3)	(5)
$q \quad , \quad r \quad \therefore \quad s$	$p, \quad r \quad \therefore \quad s$
$[(\sim P \supset Q) : \sim P] \supset Q$	$(P : \sim P) \supset Q$
T	T
T	T
T	T
T	T

In presenting this table, I have indicated the substitutions for conditions (1), (3), and (5), and have indicated by number the original argument-forms to which they correspond. Since (8) differs from (7) only in that here $\sim Q \supset P$ is substituted for *q*, while in (7) $\sim P \supset Q$ was substituted for *q*, and since $\sim P \supset Q$ and $\sim Q \supset P$ are materially equivalent, the truth-table for (8) will be precisely like the table for (7) in the arrangement of truth-values, and needs no special discussion.

In the light of our suggested criterion for consistency, this truth-table clearly illustrates the paradox to which Denise has called attention. The three expressions corresponding to the arguments of (7) are all well-known tautologies, of which the third is materially implied by the others, so that conditions (1) and (3) are satisfied, and (5) is 'implied' by them. Condition (2) is satisfied, since *p* and *q* both have the value 'T' in the first and third rows. Similarly condition (4) is satisfied, since *q*, *r*, and *s* all have the value 'T' in the second row. There is no row, however, in which *p*, *r*, and *s* are all true. They are therefore inconsistent by our criterion, and condition (6) is not satisfied.

Thus while Denise's paradoxical result fails to hold for unanalyzed *p*, *q*, *r*, and *s*, he has shown that standard substitution procedures enable us to replace these letters by expressions which seem to exhibit the paradox in a very blatant form. Indeed his substitutions have been so contrived as to make the inconsistency of *p*, *r*, and *s* inevitable, since the expressions substituted for *p* and *r* are contradictory to one another. Thus the sets of arguments (7) and (8) are of such a character as to satisfy an important additional condition, namely:

(9) p and r are *not* consistent.

This condition holds simply because of our substitutions and because of the conventions of almost any 'logic' which includes the law of non-contradiction and which permits substitutions of this sort. It is not required by conditions (1), (3), and (5) as such, nor by (2) and (4) as such, nor by all five of these taken together. But it *is* required by Denise's substitutions; any incredulous innocent who ventures to inspect the truth-table for (7) must admit that this possibility is one which he may have overlooked. Even if he is too innocent to make such an inspection, he is probably not too innocent to see that if one adds condition (9) to the others, the conclusion (6) must be false.

He must also concede that we need not use material implications in our substitutions or anywhere else to get this 'guilty' result. It is true, of course, that Denise obtained the values for q by substituting $\sim P \supset Q$ and $\sim Q \supset P$ for this variable. But we do not have to do this to make (6) come out false. We need only adop his trick of substituting $\sim p$ for r in the original set of arguments. We may then rewrite his question as follows:

> Given a sequence of valid arguments from ordinary discourse of the forms (1') $p \therefore q$, and (3') $q, \sim p \therefore s$, with (2') p and q consistent, and (4') $q, \sim p$, and s, consistent, does it follow that the argument of the form (5') $p, \sim p \therefore s$ is valid, and that (6') $p, \sim p$, and s are consistent?

The answer is obviously 'no', since p and $\sim p$ can hardly be consistent by any reasonable criterion. We need not take the trouble to turn (1'), (3'), and (5') into material implications, nor need we substitute material implications for p, q, or s; we need not even assume that the relationship which steps (1') and (3') bear to step (5') is one of material implication. Step (6') seems clearly and unparadoxically unwarranted. Here the innocent must agree with the guilty.[5]

If there *is* any paradox here, it would seem to lie in the fact that a substitution procedure can be found by which a fairly plausible sequence of argument-forms accompanied by certain attractive stipulations as to consistency can be shown to have as substitution-instances some sequences for which one of these consistency-conditions fails to hold, and that this is the case not only for 'the logic of material implication' but also for the logic of almost any kind of 'implication' one is likely to propose. If Denise's substitutions prove anything, they prove that an effect which those who espouse a 'logic of material implication' may be fortunate enough to discern should be equally discernible to those who refuse to espouse that logic, and that it is not strictly a consequence of that logic at all.[6]

It may still be worth while, however, to ask if we can construct a sequence of 'arguments from ordinary discourse' corresponding to (1') and (3') with consistency relations such as those specified in (2') and (4'), even though it is clear that their validity depends in no way upon 'the logic of material implication'. I suggest with some diffidence two such sets of arguments, in which I adopt Denise's device of making p and r contradictories.

Example I:

(1)	Helen is a lady.	p
	Therefore Helen is not a gentleman.	$\therefore\ q$
(2)	Here p and q are clearly consistent.	
(3)	Helen is not a gentleman.	q
	Helen is not a lady.	r

[5] Whether the argument-form (5') $p, \sim p \therefore s$ should also be challenged remains questionable, as does its connection with 'the logic of material implication'. See below. But Denise's innocents would apparently not go so far as to question it.

[6] My colleague Richard Cole suggests that the explanation may be even simpler, and that the paradoxical effect in due merely to the fact that one might naively expect consistency to be a transitive relation.

Therefore Helen is neither a lady nor a gentleman. $\therefore$ s

(4) Here q, r, and s are consistent.

(5) Helen is a lady. p
Helen is not a lady r
Therefore Helen is neither a lady nor a gentleman. $\therefore$ s

(6) Here p, r, and s are clearly *in*consistent.

Example II:

(1) John is a robber. p
Therefore John is a robber or Helen is a jade. $\therefore$ q

(2) Here p and q are consistent.

(3) John is a robber or Helen is a jale q
John is not a robber. r
Therefore Helen is a jade. $\therefore$ s

(4) Here q, r, and s are consistent.

(5) John is a robber. p
John is not a robber. r
Therefore Helen is a jade. $\therefore$ s

(6) Here p, r, and s are *in*consistent.

I believe that in Example I arguments (1) and (3) can be defended as valid on strictly semantical grounds. In Example II arguments (1) and (3) illustrate the well-known principles of 'addition' and 'disjunctive syllogism'. These of course can be translated respectively into arguments of the forms $p \therefore \sim p \supset q$, and $(\sim p \supset q) \cdot \sim p \therefore q$. But disjunctive syllogisms of this type have been considered valid since the time of the Stoics without benefit of such translation; and though the principle of addition clearly authorize us to make inferences which perhaps we would not choose to make 'in ordinary discourse', it can still be defended, and I think this can be done more plausibly when it is presented in its usual form.

In both examples argument (5) is more puzzling. It is obviously a very silly one which we would harly expect to find 'in ordinary discourse'. I am not at all sure that a person who is innocent of 'the logic of material implication' would be willing to concede its validity; and it is clear that any similar argument in which the two premisses are contradictories would be equally embarrassing. Of course it would be easy to defend it on the ground that the corresponding statement-form $(p . \sim p) \supset s$ is a tautology; but this might be considered too abject a capitulation to 'the logic of material implication'. One might also defend it by insisting flatly that from a pair of contradictory premisses any conclusion may be validly derived. Whether one can justify this principle without employing material implication either explicitly or in disguise is questionable. One can do so explicitly by the familiar procedure of pointing out that given any pair of contradictories, one of them must be false, and that a false proposition materially implies any proposition whatsoever. And of course one might also try to establish this principle by the equally familiar use of addition and a disjunctive syllogism, which, as we have seen, are translatable into argument-forms employing material implications. Presumably one does not have to call these 'material implications' or represent them by horseshoes or arrows; their 'logic' will smell the same. But which version provides the real disguise? And it is really a disguise at all? Furthermore, if one seeks to validate argument (5) by asserting that the conclusion follows simply because the premisses are contradictory, one could just as well use this most accommodating principle again to validate anything one chooses, including the really shocking conclusion that p, r, and s *are* consistent after all, and that p and $\sim p$ really *aren't* contradictory. When the sauce is so highly seasoned, it is not easy to distinguish the guilt of the gander from the innocence of the goose.

Of course, as Denise takes pains to point out, no logician who is a gentleman, whether innocent or guilty, has any business to be *asserting* contradictory premisses. If he arrives at them by substitution, there may well be something ungentlemanly in his substitution procedures.

RESUMEN

En un artículo profundo y sugerente,[1] Theodore C. Denise ha llamado la atención sobre un nuevo problema en la teoría de la implicación material.

> Dada en el lenguaje ordinario una secuencia de argumentos válidos de la forma $p \therefore q$, y $q, r \therefore s$, siendo las proposiciones p y q consistentes, al igual que q, r y s, ¿se sigue que el argumento de forma $p, r \therefore s$ es válido y que las proposiciones p, r y s son consistentes? Creo que no dudaríamos en dar una respuesta afirmativa si desconocemos la lógica de la implicación material; pero si la aceptamos, decimos solamente que el argumento es válido y que sus proposiciones componentes pueden o no formar un conjunto consistente... Para algunos, una lógica tan "paradójica" sólo puede emplearse con recelo (p. 62).

Al plantear su problema, Denise ha evitado cuidadosamente el hacer uso explícito de la implicación material. Pero se puede ver que deben tomarse como dadas las siguientes cuatro condiciones:

(1) $p \supset q$;
(2) p y q son mutuamente consistentes;
(3) $(q \,.\, r) \supset s$;
(4) q, r y s son mutuamente consistentes.

El problema es ver si se sigue que

(5) $(p \,.\, r) \supset s$; y
(6) p, r y s son mutuamente consistentes.

La condición (5) es derivable de las condiciones (1) y (3) juntas, y es materialmente implicada por ellas. Respecto a las otras condiciones, Denise no logra establecer ningún criterio de consistencia. Se podría sugerir que un conjunto de proposiciones es 'consistente' si y sólo si la columna correspondiente a su conjunción en una tabla de verdad común tiene al menos una 'T' o, más simplemente, si y sólo si hay al menos un renglón en la tabla de verdad que tenga una 'T' para cada proposición del conjunto. Pero Denise no permite que p, q, r y s permanezcan sin ser analizadas, y trata de establecer su paradoja de dos maneras. Su primer procedimiento

[1] "Material Implication Re-examined", *Mind* XXI, No. 281, January 1962, pp. 62-68.

consiste en sustituir P por p, $(\sim P \supset Q)$ por q, $\sim P$ por r, y Q por s en las tres formas argumentales, obteniendo el siguiente conjunto de argumentos:

(7) $P \therefore \sim P \supset Q$; $\sim P \supset Q, \sim P \therefore Q$; y $P, \sim P \therefore Q$.

Luego vuelve a las tres formas argumentales originales, y sustituye P por p, $(\sim Q \supset P)$ por q, $\sim P$ por r, y Q por s, obteniendo así el nuevo conjunto de argumentos:

(8) $P \therefore \sim Q \supset P$; $\sim Q \supset P, \sim P \therefore Q$; y $P, \sim P \therefore Q$.

Reemplazando los argumentos de (7) por sus correspondientes expresiones implicativas $P \supset (\sim P \supset Q)$; $[(\sim P \supset Q) . \sim P] \supset Q$; y $(P . \sim) \supset Q$, se puede formar una tabla de verdad* que muestra la paradoja a que alude Denise. Las tres expresiones implicativas anteriores son tautologías, siendo la tercera materialmente implicada por las otras. La condición (2) es satisfecha, puesto que p y q tienen el valor 'T' en los renglones 1o. y 3o. También es satisfecha la condición (4), ya que q, r y s tienen el valor 'T' en el 2o. renglón. Sin embargo, no hay ningún renglón en que p, r y s sean todas verdaderas. Son, por tanto, inconsistentes según el criterio establecido, y no es satisfecha la condición (6). (La tabla de verdad de (8) es como la de (7), y no necesita de una discusión especial). Denise ideó sus sustituciones de tal modo que p y r fuesen mutuamente contradictorias. Así, los conjuntos de argumentos (7) y (8) satisfacen una importante condición adicional, a saber:

(9) p y r *no* son consistentes.

Pero si se añade la condición (9) a las demás, la conclusión (6) debe ser falsa. Por otra parte, para hacer falsa a (6), basta con el truco de sustituir $\sim p$ por r en el conjunto original de argumentos, y no es necesario sustituir las implicaciones materiales $\sim P \supset Q$ y $\sim Q \supset P$ por q, quedando reformulada la cuestión como sigue:

> Dada en el lenguaje ordinario una secuencia de argumentos válidos de las formas (1') $p \therefore q$, y (3') $q, \sim p \therefore s$, con (2') p y q consistentes, y (4') $q, \sim p$, y s consistentes, ¿se sigue que el argumento de la forma (5') $p, \sim p \therefore s$ es válido y que (6') $p, \sim p$, y s son consistentes?

En esta reformulación no es necesario tomarse la molestia de transformar (1'), (3') y (5') en implicaciones materiales, ni de sus-

* Cfr. pp. 114-115.

tituir implicaciones materiales por *p*, *q*, o *s*, ni de asumir que la relación de (1') y (3') con (5') es de implicación material. Si las sustituciones de Denise prueban algo, prueban que un resultado que puede ser distinguido por aquellos que defienden una 'lógica de la implicación material' debería ser igualmente distinguible para quienes se rehusan a defender esa lógica, y que dicho resultado no es estrictamente una consecuencia de esa lógica. Además, resulta difícil creer que una persona que desconozca 'la lógica de la implicación material' esté dispuesta a aceptar la validez de algún caso de sustitución del argumento (5'). Desde luego, sería fácil defender tal argumento sobre la base de que su correspondiente forma proposicional $(p \,.\, {\sim} p) \supset s$ es una tautología; pero esto podría considerarse como una capitulación demasiado abyecta para 'la lógica de la implicación material'. Podría también decirse llanamente que de un par de premisas contradictorias puede derivarse válidamente cualquier conclusión. Pero es discutible si este principio puede ser justificado sin emplear la implicación material explícita o disfrazadamente. Si uno trata de hacer válido el argumento (5) afirmando que la conclusión se sigue simplemente porque las premisas son contradictorias, igualmente podría usar este comodísimo principio para hacer válida cualquier cosa que escoja, incluyendo la conclusión realmente chocante de que, después de todo, *p*, *r* y *s son* consistentes, y de que *p* y $\sim p$ *no son* realmente contradictorias.