SKOLEM'S PARADOX AND PLATONISM*

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1. In a paper recently published in this journal, F. Miró Quesada claims that no contemporary philosophy of mathematics is in the least satisfactory.[1] Of course this is not a new claim and is in many respects plausible, particularly if intended in the sense that no such philosophy has yet reached an adequate stage of development. But that is not what Miró Quesada means. In his opinion all philosophies of mathematics hitherto elaborated, including the most important of them, platonism and intuitionism, are confronted with difficulties which make them already untenable. Any examination of his argument necessarily requires a precise formulation of these philosophies, which cannot be given here. We shall confine ourselves to the argument against platonism; this restriction does not signify any special preference but only the fact that, at present, the platonist position is certainly more developed and in general better known than the intuitionistic one.

Miró Quesada rightly points out that the platonist position is in no way invalidated by Gödel's first incompleteness theorem. As is well known, platonism in its set theoretic version asserts that mathematics is about objects external to us which are taken to constitute a hierarchy $\langle V_\alpha, \epsilon_\alpha \rangle$,

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[1] F. Miró Quesada, "La objeción de Rieger y el horizonte de la ontología matemática", *Crítica*, No. 5, vol. II (1968), pp. 55-70.

for any ordinal α, the so-called *type structure*, where $V_0 = \emptyset$; $V_\alpha = \bigcup_{\beta<\alpha} P(V_\beta)$ for $\alpha > 0$, and ϵ_α is the restriction of the membership relation to V_α, i.e. such that $x \epsilon_\alpha$ if and only if $x, y \epsilon V_\alpha$ and $x \epsilon y$. Members of V_α, for some α, are called *sets*.[2] On account of the fact that sets exist independently of our understanding of them, the impossibility of describing all their properties in such a comparatively poor language as the first order language of set theory is not surprising.[3] Incidentally Gödel's first incompleteness theorem does not affect the intuitionistic position either, since the latter is concerned with mental operations *(constructions)* for which not all properties are expected to be decidable.

2. For the platonist position Miró Quesada however attaches an invalidating rôle to the Löwenheim-Skolem theorem, reviving an argument originally due to Skolem. According to that theorem any denumerable set of formulae of a first order language (e.g. the first order Zermelo-Fraenkel axioms ZF^1 of set theory) which has an infinite model has a denumerable model.[4] Now, let x be the set whose existence

[2] For a formulation of platonism, cf. K. Gödel, "Russell's mathematical logic", in *The philosophy of Bertrand Russell*, ed. P. A. Schilpp, Northwestern University Press, Evanston, Ill., 1944, pp. 123-153; "What is Cantor's continuum problem?", in *The philosophy of mathematics*, ed. P. Benacerraf and H. Putnam, Prentice Hall, Englewood Cliffs, N. J., 1964, pp. 258-273. For a systematic analysis, see G. Kreisel and J. L. Krivine, *Elements of mathematical logic (model theory)*, North-Holland, Amsterdam, 1967, Appendix II, Part A.

[3] The first order language of set theory is a (first order) language whose variables $x, y, \dots$ range over sets and whose only non-logical predicate symbol ϵ stands for the membership relation. The second order language of set theory includes also second order variables $X, Y, \dots$.

[4] The Zermelo-Fraenkel axioms of set theory, in their original form, present themselves as (second order) axioms ZF^2 expressed in the second order language of set theory. Specifically this is the case for the axioms of foundation, comprehension and replacement. Cf. E. Zermelo, "Über Grenzzahlen und Mengenbereiche", *Fundamenta Mathematicae*, vol. 16 (1930), pp. 29-47. The corresponding first order axioms ZF^1 are obtained by replacing the second order axioms by first order axiom schemata, expressed in the first order language of set theory. See R. Montague, "Set theory and higher order logic", in *Formal systems and recursive functions*, ed. J. N. Crossley and M. A. E. Dummett, North-Holland, Amsterdam, 1965, pp. 131-148; G. Kreisel, "A survey of proof theory", *The Journal of Symbolic Logic*, vol. 33 (1968), pp. 321-388, § 4.

is guaranteed by the axiom of infinity. By the power set axiom there is a set y including as members all subsets of x, and by Cantor's result (a theorem of ZF^1) there is no $1 - 1$ correspondence between x and y. Considering the denumerable model of ZF^1 given by the Löwenheim-Skolem theorem we get a contradiction, the so-called *Skolem's paradox.*

According to the explanation originally proposed by Skolem, the paradox arises from attributing an absolute character to set theoretic notions. In the denumerable model y is not the set of all subsets of x but only the set of all subsets of x belonging to the model. Since both x and y are, of course, denumerable there exists a $1 - 1$ correspondence between x and y. But the correspondence (a set of ordered pairs) is not a member of the denumerable model because Cantor's theorem is valid in any model of ZF^1. Thus in the denumerable model there is no $1 - 1$ correspondence between x and y, i.e. the set y is nondenumerable in the model. Hence the notions of set of all subsets, $1 - 1$ correspondence, nondenumerability etc., are relative to a particular model of ZF^1.[5]

In such an explanation two different aspects occur which it will be worthwhile to emphasize. First of all it shows that use of the expression "paradox" is in this case improper. In fact we are not faced here by a contradiction implicit in ZF^1 but only by a consequence of the false assumption that a first order axiomatization can uniquely characterize set theoretic concepts. This part of the explanation is perfectly legitimate and offers no problem at all. On the other hand, by assigning a privileged rôle to first order axiomatizations, the explanation regards their inadequacy as evi-

[5] Cf. T. Skolem, "Some remarks on axiomatized set theory", in *From Frege to Gödel (a source book in mathematical logic, 1879-1931)*, ed. J. van Heijenoort, Harvard University Press, Cambridge, Mass., 1967, pp. 291-301; "Sur la portée du théorème de Löwenheim-Skolem", in *Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques, 6-9 décembre 1938*, ed. F. Gonseth, Leemann, Zurich, 1941, pp. 25-47.

dence against platonism. In other words the argument is as follows: first order axiomatizations are unable to characterize uniquely set theoretic notions, hence there is no particular privileged set theoretic notion.

Clearly this part of the explanation is hardly justified. First of all it presupposes that set theoretic notions are *implicitly defined* by the axioms. Actually, from a platonist point of view, axioms are intended only to describe given notions; thus their inadequacy would be no evidence against platonism in view of the above argument mentioned in connection with Gödel's first incompleteness theorem. Secondly, it seems to disregard the fact that from a platonist viewpoint first order axiomatizations play no special rôle, since the notions of first order and higher order consequence are defined in terms of the same basic set theoretic notions.

3. The thesis that Skolem's paradox necessarily implies that all set theoretic notions are relative is justified only on the basis of an *abstract* conception of mathematics rejecting the existence of an intuitive basic notion of set which must be analysed in order to determine its properties which are then formulated by suitable axioms. The notion of set is implicitly defined by the first order axioms of ZF^1 just as the notion of group, ring, field or vector space is defined by the first order axioms of the theory of groups, rings, fields, vector spaces respectively. In other words set theory is an abstract theory in the sense of algebraic theories.

Consequently although Gödel's first incompleteness theorem establishes the existence of statements of the first order language of set theory restricted to $\langle V_\omega, \epsilon_\omega \rangle$ which are true in this structure and are not theorems of ZF^1, this is no evidence of the inadequacy of ZF^1, still less of first order axiomatizations in general. Simply they should not be considered as set theoretic truths. Similarly the non unique definability of infinite structures, including $\langle V_\omega, \epsilon_\omega \rangle$, by formulae of the first order language of set theory (a consequence of the Löwenheim-Skolem theorem) is only a distinctive feature which *happens* to be true of first order axiomatizations.

To realize the implausibility of this position let us simply note that the notion of structure is defined directly in terms of the basic set theoretic notions. Hence it would be circular to state that any set theoretic notion is relative to a particular structure which is a model of ZF^1. In other words the alleged analogy of algebraic notions to the notion of set, which engendered it, overlooks the fact that the former are derived whereas the latter is basic. Furthermore the subordination of higher order to first order axiomatizations takes no account of the fact that, if no basic notion of set is accepted, not only the notion of second order consequence but also that of first order consequence will be relative to the specific model of ZF^1 considered. Indeed, as mentioned above, both are defined in terms of the same basic set theoretic notions.

The abstract conception does not permit the use of second order notions on the grounds that it would involve presupposing the concept of set in axiomatizing that concept.[6] In fact this is the basic reason why the conception is confined to first order axiomatizations. Platonistically, however, the notion of set is *given* by the type structure, and there is no circularity involved in using a given notion to state (some of) its properties. Also, from an historical point of view, the position seems to ignore the fact that axiomatizations of abstract (algebraic) theories were never meant to formulate properties of intuitive basic notions. The existence of non isomorphic models for algebraic theories not only fails to provide new information about the properties of the underlying notions, but even constitutes a *prerequisite* for them to satisfy![7]

[6] See, e.g., A. Mostowski, "O niektórych nowych wynikach meta-matematycznych dotyczacych teorii mnogosci", *Etudia logica*, vol. 20 (1967), pp. 99-112. Cf. p. 110.

[7] In axiomatizations of abstract theories the following circumstances are equally undesirable. First of all, trivially, if the axioms have no model, then they are vacuously valid. On the other hand, if all their models are isomor-

In the specific case of Quesada's paper one could also object that in refuting the platonist position it is rather inconsistent to attach any importance to the Skolem's paradox while denying it to Gödel's first incompleteness theorem. From a platonist point of view the former is certainly no more disturbing than the latter. In fact as is hardly surprising it turns out that no complete characterization of the notion of set can be obtained by means of axioms like ZF^1 which are expressed in the first order language of set theory; equally it is not surprising that any denumerable set of axioms formulated in that language and satisfied by a segment of the type structure including $\langle V_\omega, \epsilon_\omega \rangle$ has a denumerable model. As only first order formulae are involved, at most a denumerable set of subsets of a given set will be definable. This is the case for ZF^1 where the axioms provide only a denumerable infinity of operations for building new sets, hence the possibility of a denumerable model of ZF^1 is easily explained. In fact this is explicitly shown in Gödel's socalled constructible model of ZF^1.[8] Consequently Skolem's paradox and Gödel's first incompleteness theorem no more provide evidence against platonism than the latter assigns a privileged rôle to first order axiomatizations.

4. To define infinite structures uniquely it is necessary to appeal to higher order axiomatizations. This is well known, for instance, in case of arithmetic, i.e. the structure $\langle N, S \rangle$, for N the set of natural numbers and $S \subset N \times N$ the successor relation on N; which is isomorphic to $\langle V_\omega, \epsilon_\omega \rangle$, or analysis, i.e. the structure $\langle R, Q, < \rangle$, for R the set of real numbers, Q the set of rational numbers (a denumerably

phic, any property of one holds of the other. Actually we are interested only in models sharing a *specific* property, i.e., of being a group, a ring, a field, a vector space, etc.

[8] Cf., e.g., A. Mostowski, *Constructible sets with applications*, Panstwowe Wydawnictwo Naukowe, Warszawa, and North-Holland, Amsterdam, 1969, Ch. 3-6.

dense subset of R) and $< \subset R \times R$ the natural order relation on R, which is isomorphic to $\langle V_{\omega+1}, \epsilon_{\omega+1} \rangle$:[9]

A more general problem is to give a unique second order characterization of the least segment of the type structure, which is a model of ZF^2, $\langle V_{\pi_0}, \epsilon_{\pi_0} \rangle$; for π_0 the first inaccessible ordinal.[10] In fact it can be shown that a unique definition of $\langle V_{\pi_0}, \epsilon_{\pi_0} \rangle$ exists: there is a formula $ZF^2_{\pi_0}(S)$ of the second order language of set theory such that, if $ZF^2_{\pi_0}(S)$ is true, where $S = \langle U, E \rangle$, $U \neq \emptyset$ and $E \subset U \times U$, then S is isomorphic to $\langle V_{\pi_0}, \epsilon_{\pi_0} \rangle$.[11] This has an important consequence. For any formula A of the second order language of set theory, let $A(S)$ be the defining formula of: $S = \langle U, E \rangle$ is a model of A, and $ZF^2_{\pi_0} \models A$ the defining formula of: A is a second order consequence of $ZF^2_{\pi_0}$. $A(S)$ is obtained from A by replacing each atomic formula $x \in y$ by $\langle x, y \rangle \in E$, $X(\langle x_1, \ldots, x_n \rangle)$ by $\langle x_1, \ldots, x_n \rangle \in X$ and each quantifier (Qx) by $(Qx \in U)$, (QX) by $(QX \subset U)$. $ZF^2_{\pi_0} \models A$ is the formula $(\forall S)\ (ZF^2_{\pi_0}(S) \rightarrow A(S))$. Since there exists, up to isomorphism, a single S such that $ZF^2_{\pi_0}(S)$, for that S we have $ZF^2_{\pi_0} \models A$ if and only if $A(S)$. Now, for

9 See, e.g., G. Kreisel and J. L. Krivine, *loc. cit.,* Ch. 7, Ex. 1. The existence of a unique definition is to be understood in the precise sense: there is a formula of the second order language of set theory whose class of principal models contains, up to isomorphism, a single element.

10 An inaccessible ordinal is a cardinal π such that: (i) $\pi > \omega$, (ii) $\alpha < \pi$ implies $2^\alpha < \pi$, for any cardinal α, (iii) sup $\alpha_i < \pi$, for any family $(\alpha_i)_{i \in I}$ of cardinals $< \pi$ indexed by a cardinal $I < \pi$.

11 The basic idea of the proof dates back to E. Zermelo, *loc. cit.* In this connection the following statement by Shepherdson is significant: "Results essentially equivalent [...] were obtained by Zermelo [...] although in an insufficiently rigorous manner. He appeared to take no account of the relativity of set-theoretical concepts pointed out by Skolem [...], assuming that such concepts as sum set, power set, cardinal number, etc., had an absolute significance" (J. C. Shepherdson, "Inner models for set theory II", *The Journal of Symbolic Logic,* vol. 17 (1952), p. 227). Clearly Shepherdson appears to take no account of the fact that Zermelo's proof refers to ZF^2 which Skolem's relativity does not apply to. In fact that is exactly what the argument is aimed at proving!

any A such that $A \leftrightarrow A(S)$, one of A and $\neg A$ is satisfied by S. Hence $ZF^2_{\pi_0} \models A \vee ZF^2_{\pi_0} \models \neg A$, in other words A is decided by $ZF^2_{\pi_0}$.

As an application consider the continuum hypothesis CH: for any $x \subset P(\omega)$ there is a $1 — 1$ correspondence between x and either $P(\omega)$ or y, for some $y \subset \omega$. $1 — 1$ correspondences between subsets of $P(\omega)$ are members of $P(P(\omega))$, which is certainly contained in any S such that $ZF^2_{\pi_0}(S)$; in fact there is a $1 — 1$ correspondence between $P(P(\omega))$ and $V_{\omega+2}$. Hence $CH \leftrightarrow CH(S)$ and CH is decided by $ZF^2_{\pi_0}$.

5. From the point of view of the abstract conception of mathematics the above results can be interpreted as follows. Let M by any model of ZF^1. (1º) The formula $ZF^2_{\pi_0}(S)$ defines the structure $\langle V_{\pi_0}, \epsilon_{\pi_0} \rangle$ uniquely relative to M, say $\langle V_{\pi_0}, \epsilon_{\pi_0} \rangle_M$: Thus if $M \neq M'$ then $\langle V_{\pi_0}, \epsilon_{\pi_0} \rangle_M$ and $\langle V_{\pi_0}, \epsilon_{\pi_0} \rangle_{M'}$ are not isomorphic. (2º) Let S be such that $ZF^2_{\pi_0}(S)$. Then for any A; such that $A \leftrightarrow A(S)$, $ZF^2_{\pi_0} \models A \vee ZF^2_{\pi_0} \models A$; for $\models$ the second order consequence relation relative to M say $\models_M$. Thus if $M \neq M'$ it may be the case that $ZF^2_{\pi_0} \models_M A$ and $ZF^2_{\pi_0} \models_M \neg A$, or viceversa; i.e. we get different decisions for different models. Consequently the above results cannot be used as evidence that the axioms of ZF^2 single out a basic structure (giving the notion) ot set unless one assumes a priori that such a structire does exist.

On the other hand they cannot be used either to refute the existence of a basic notion of set. The essential circularity of Skolem's argument against platonism consists in the fact that, by rejecting such an existence initially and considering the axioms of ZF^1 as a definition of set, it takes the limitations of ZF^1 as evidence that no basic notion of set exists.

From a platonist point of view the main interest of unique definitions stems from the fact that they provide a direct reduction of the defined structure to the primitive notions of (the language of) the definition. Thus the unique definitions of $\langle N, S \rangle$ and $\langle R, Q, < \rangle$ reduce the notion of natural number of real number respectively to the notion of set. Second order notions are essential here although first order methods are more fruitful at present in applications such as the independence proofs because more is known about first order than about second order consequence. For instance *CH* is decided by $ZF^2_{\pi_0}$ but nobody knows which way it is decided, i.e. which of *CH* and $\neg CH$ is valid in $\langle V_{\pi_0}, \epsilon_{\pi_0} \rangle$. An important problem is whether strong axioms of infinity are all we need to make $\models$ fruitful or whether an expansion of the second order language of set theory is required.

6. The unique definability of certain segments of the type structure does not extend to the whole structure. In fact suppose a unique second order definition $ZF^2_\Lambda(S)$ of the type structure exists in the second order language of set theory. Then there is an isomorphism of the structure defined by $ZF^2_\Lambda(S)$ to the type structure, and hence a $1-1$ correspondence between a particular set and the whole universe of sets; but this cannot be established by the definition of set.[12] Confronted with the problem of giving a unique definition of the type structure, we realize that the second order language of set theory leaves us in the lurch. Actually, not only is there no unique definition, but no definition whatsoever of the whole type structure could exist in that language. This is a consequence of Tarski's theorem on truth: the set of all statements of the second order language of set theory which are true in the type structure is not definable in terms of that language. Thus, if we want to consider such entities as the type structure in any sense like a single mathematical object, then there seems to be no other way than by expanding the

[12] The same applies to the notion of ordinal with respect to the language of the theory of ordinals. Of course this has nothing to do with the existence of a unique second order definition of the notion of ordinal in the language of set theory (pointed out by G. Kreisel and J. L. Krivine, *loc. cit.*, pp. 169-170) which only provides a direct reduction of the notion of ordinal to the notion of set, given the latter.

language of set theory by symbols for new primitive notions.[13]

The intrinsic limitations of the language of set theory are a consequence of considering only set theoretic properties rather than properties of the most general kind (or *concepts*), possibly undefined for certain singular points.[14] A natural expansion of the language of set theory would include variables for concepts with a predicate symbol for the binary relation: *x presoupposes y*, and possibly others. An essential test for the supposed new primitives would be: do they extend the scope of our understanding of our mathematical experience? More specifically what is really sought here is a unique definition of the primitive notions of the second order language of set theory in terms of the new notions.

The new primitives would lead to a far better approximation to what platonist objects *are* than the notion of set. Of course they would provide a new explanation of paradoxes. The property $C(X)$ of being a concept which applies to a concept X if and only if X does not apply to itself is undefined for the argument C since the application of C to C presupposes C to be conceived. This must not be confused with the Poincaré-Russell vicious circle principle, at least in its current constructive version.[15] For conceiving can hardly be supposed to have any connection at all with the existence of a definition of a certain elementary kind, reducing abstract existential assumptions to purely arithmetic ones.

[13] Of course only non trivial expansions are meant here. For instance validity with respect to principal models of higher (finite) order languages of set theory is reducible to validity in the principal models of the second order languages of set theory. See, e.g., G. Kreisel and J. L. Krivine, *loc. cit.*, Ch. 7, Th. 1.

[14] On concepts cf., e.g., the remarks in K. Gödel, "Russell's mathematical logic", *cit.*

[15] Cf. S. Feferman, "Systems of predicative analysis", *The Journal of Symbolic Logic,* vol. 29 (1964), pp. 1-30. For different versions of the vicious circle principle, see K. Gödel, *loc. cit.*

RESUMEN

En un trabajo recientemente publicado en esta revista, Francisco Miró Quesada afirma que todas las filosofías de la matemática elaboradas hasta ahora, incluyendo el platonismo y el intuicionismo, resultan insostenibles. Discutimos aquí solamente su argumento en contra del platonismo por la razón de que en la actualidad la posición platónica es generalmente mejor conocida que la intuicionista.

La prueba que ofrece Miró Quesada depende casi por completo de la llamada *Paradoja de Skolem.* Su argumento es, de hecho, el argumento de Skolem: la paradoja muestra que las axiomatizaciones de primer orden no son aptas para caracterizar de manera única nociones de teoría de conjuntos y, por ende, que no hay ninguna noción particular de teoría de conjuntos que resulte privilegiada. Obviamente, el argumento presupone que estas nociones están implícitamente definidas por los axiomas. Ahora bien, desde un punto de vista platónico, los axiomas tienen por objeto solamente describir nociones dadas; por tanto, su inadecuación no constituiría prueba alguna contra el platonismo. Si los conjuntos existen independientemente de que los comprendamos, no es de extrañar la imposibilidad de describir todas sus propiedades en un lenguaje tan pobre como el lenguaje de primer orden de teoría de conjuntos. En segundo lugar, el argumento parece hacer caso omiso del hecho de que para una postura platónica las axiomatizaciones de primer orden no tienen un papel especial que jugar dado que las nociones de consecuencia de primer orden, y de órdenes superiores, se definen en términos de las mismas nociones básicas de teoría de conjuntos.

La idea subyacente al argumento de Skolem es la concepción abstracta de las matemáticas que no acepta la existencia de una noción intuitiva básica de conjunto y considera a la teoría de conjuntos como una teoría abstracta, en el sentido de las teorías algebraicas: la noción de conjunto se halla implícitamente definida por los axiomas de primer orden de Zermelo-Fraenkel así como la noción de grupo está implícitamente definida por los axiomas de primer orden de la teoría de grupos. Así pues, cualquier noción de teoría de conjuntos es relativa a una estructura dada la cual es un modelo de los axiomas. Desde esta perspectiva tenemos que la definibilidad no única de estructuras infinitas mediante fórmulas del lenguaje de primer orden de teoría de con-

juntos es solamente un rasgo distintivo que resulta verdadero de las axiomatizaciones de primer orden.

A manera de poner en evidencia la implausibilidad de esta concepción, notemos simplemente que resulta circular asumir que las nociones de teoría de conjuntos son relativas a estructuras dadas, pues, la noción de estructura está definida a su vez en términos de las nociones básicas de teoría de conjuntos. En segundo lugar, esta postura ignora el hecho de que las axiomatizaciones de teorías algebraicas abstractas nunca se hicieron con el objeto de formular propiedades de nociones intuitivas básicas. La existencia de modelos no isomórficos para las teorías algebraicas no sólo no proporciona nueva información acerca de las propiedades de las nociones subyacentes, sino que incluso constituye un prerequisito que aquéllos deben satisfacer.

La circularidad básica del argumento de Skolem en contra del platonismo consiste en el hecho de que al rechazar la existencia de una noción básica de conjunto y considerar los axiomas de primer orden de Zermelo-Fraenkel como una definición de conjunto, toma las limitaciones de los axiomas como una prueba de que no existe una noción básica de conjunto. Por otro lado, si se acepta la existencia de una noción básica de conjunto, hay una fórmula en el lenguaje de segundo orden de teoría de conjuntos que define de manera única hasta el más pequeño segmento de la llamada estructura tipo, la cual es un modelo de los axiomas de primer orden de Zermelo-Fraenkel. Sin embargo, tal definibilidad única no se extiende a toda la estructura tipo: esta es una consecuencia del teorema de Tarski sobre la verdad. Ahora bien, desde un punto de vista platónico el interés primordial en las definiciones únicas reside en el hecho de que proporcionan una reducción directa de la estructura definida a las nociones primitivas del lenguaje de la definición. Por tanto, si queremos considerar entidades tales como la estructura tipo como un sólo objeto matemático, no hay más que una manera: expandiendo el lenguaje de teoría de conjuntos mediante la introducción de símbolos para nuevas nociones primitivas tales como la noción de propiedad intensional.